Fundamentals of Electromagnetics

Magnetic Field, Current, and Inductance

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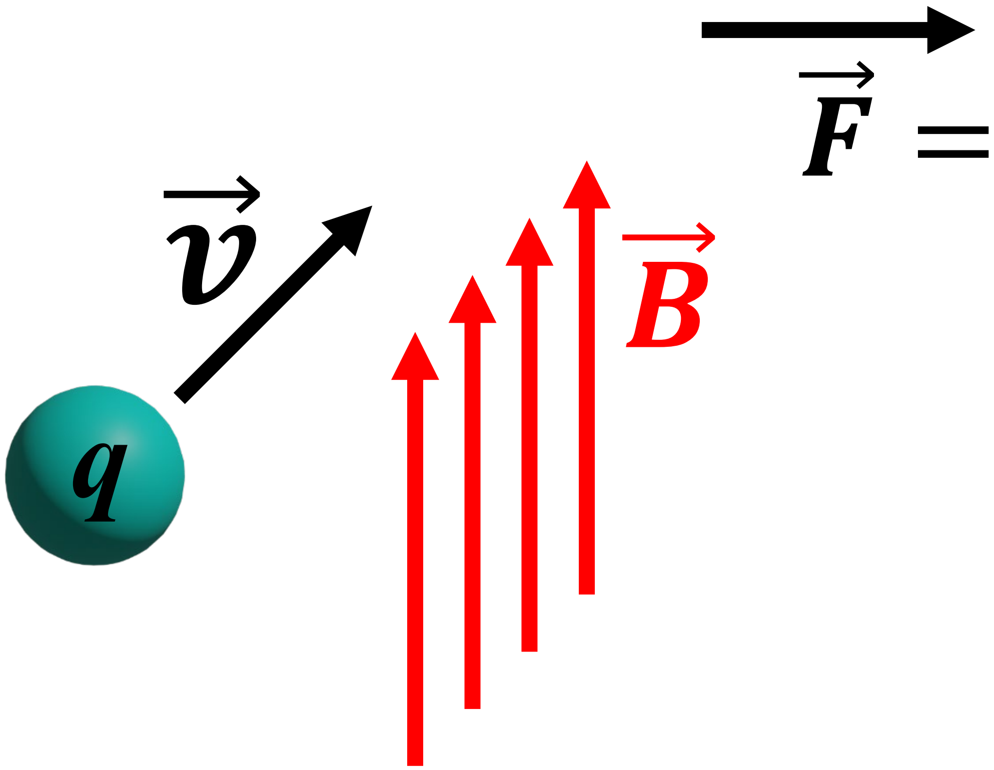
Topics

- **Magnetic Field, Force, and Charge**
- **Magnetic Field Intensity and Magnetic Flux Density**
- **Ampère's Law**
- **Permeability**
- **Faraday's Law of Electromagnetic Induction**
- **Inductance**
- **(Virtual) Demonstration: Inductive Coupling**
- **Magnetic Field, Current, and Inductance**

- ***Material taken from "Fundamentals of Electromagnetics" video series***
  - *Publicly available on YouTube; search for above title*
  - *Direct playlist link:*
    - https://youtube.com/playlist?list=PLtrpQ-gPvnJn2r9Mw49jij7Ky0mb6RJYF&si=UxEKqVRgsR9w6nZ7

Magnetic Force and Charge

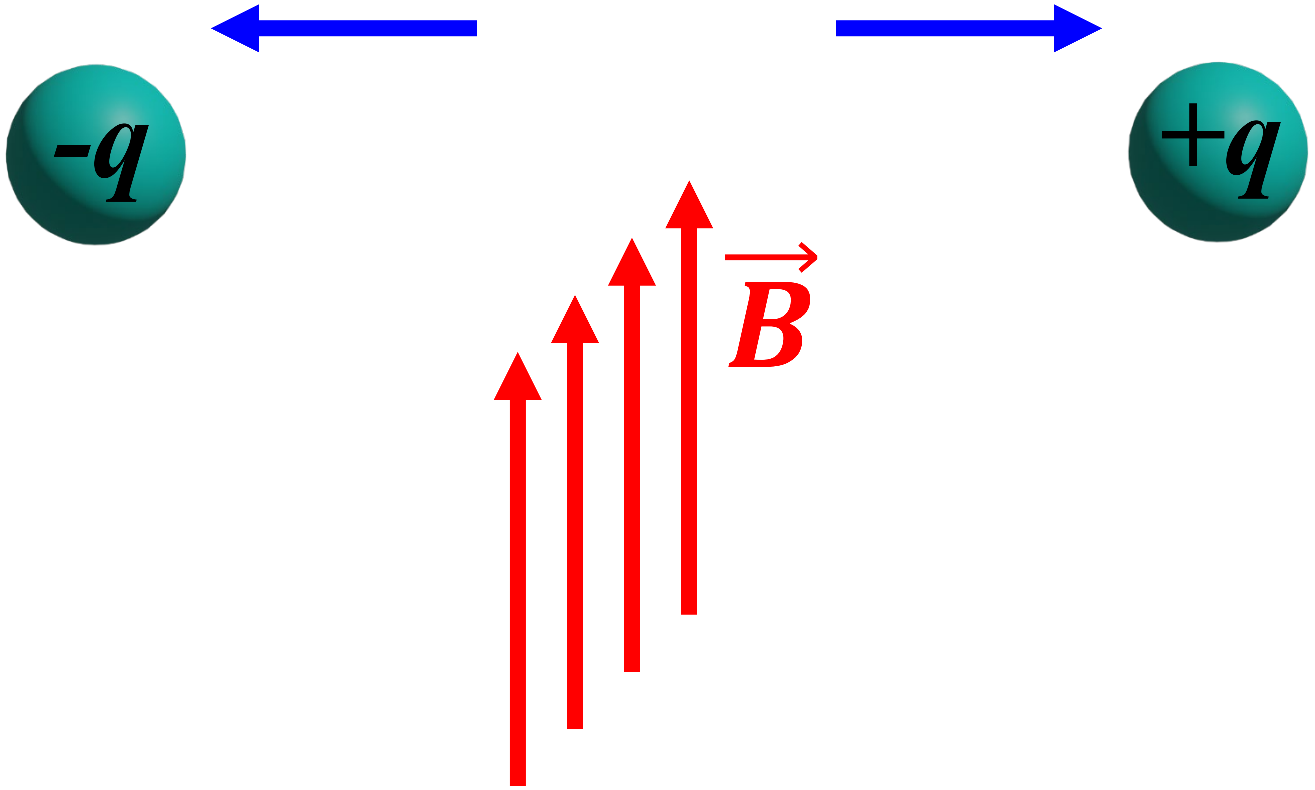


$$\vec{F} = q\vec{v} \times \vec{B}$$

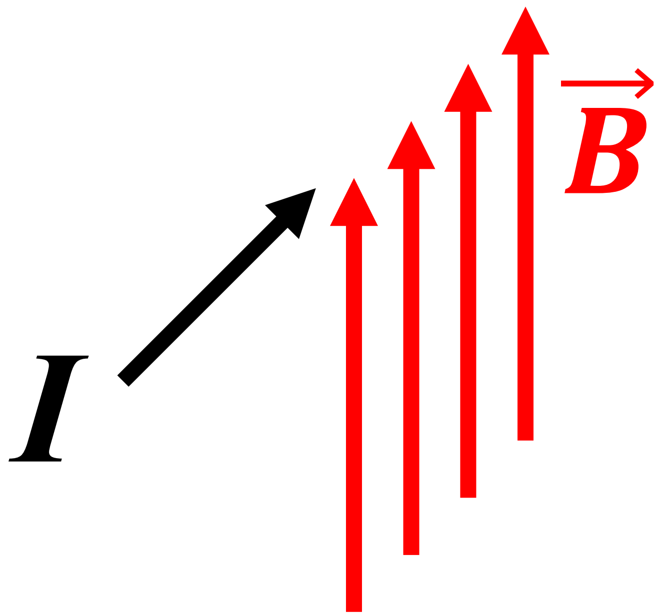
Magnetic Force and Charge (cont.)

Negative charges move in opposite direction of magnetic force

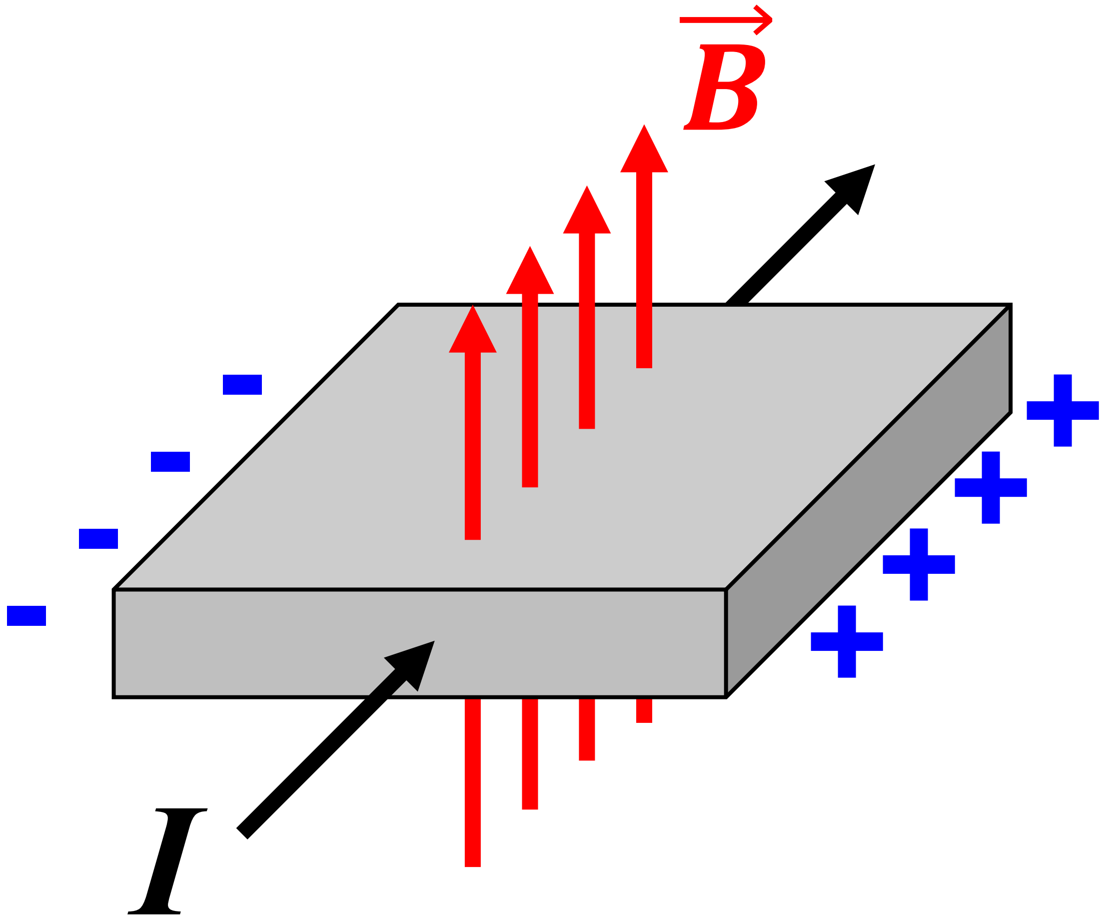
Positive charges move in direction of magnetic force



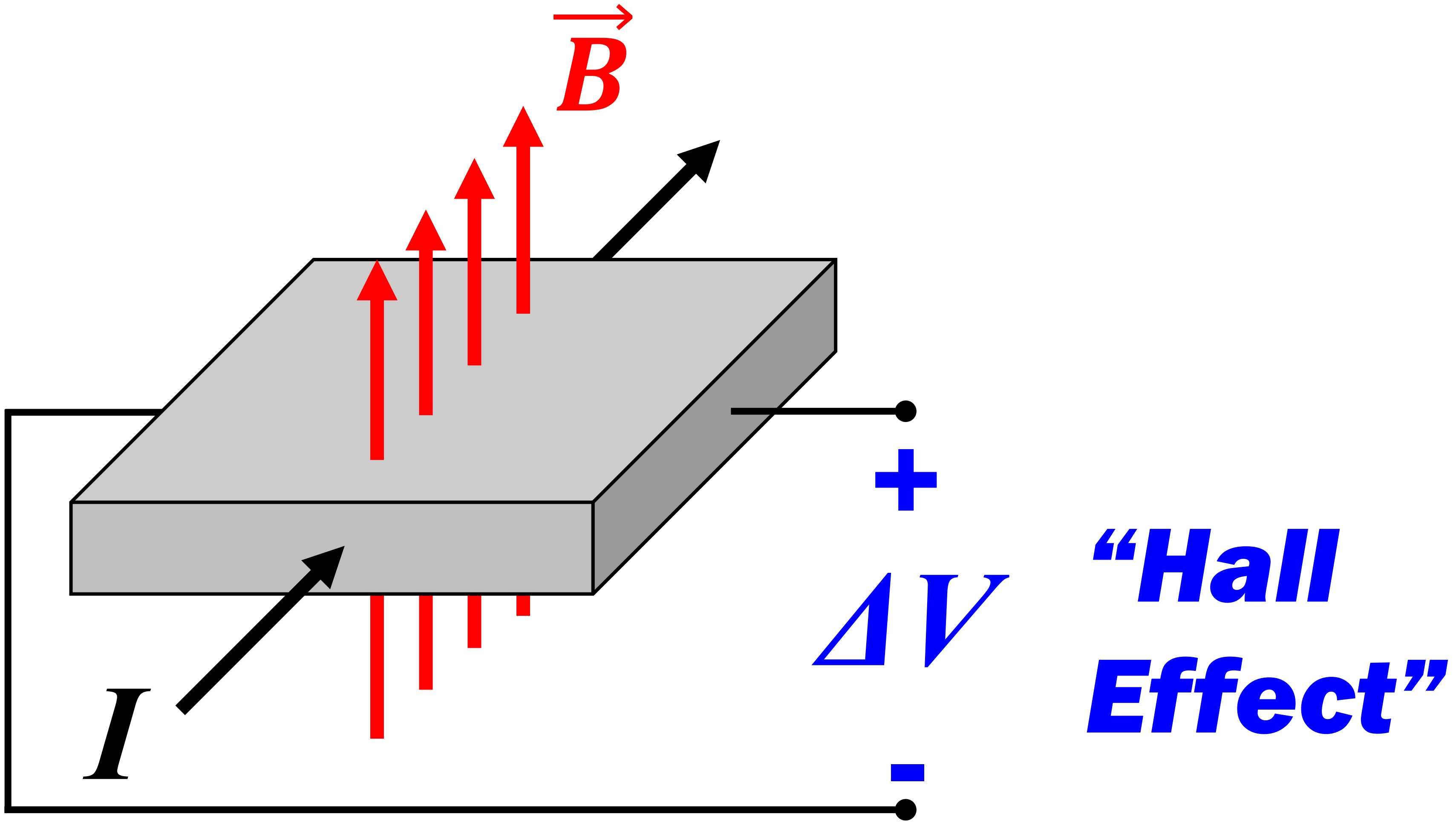
Moving Charge = Current



Current and Charge Separation



Current and Charge Separation: Hall Effect



Moving Machinery (motors, etc.)

If the magnetic force can move charges, it can move the conductors carrying those charges…

→ ***MOTORS***

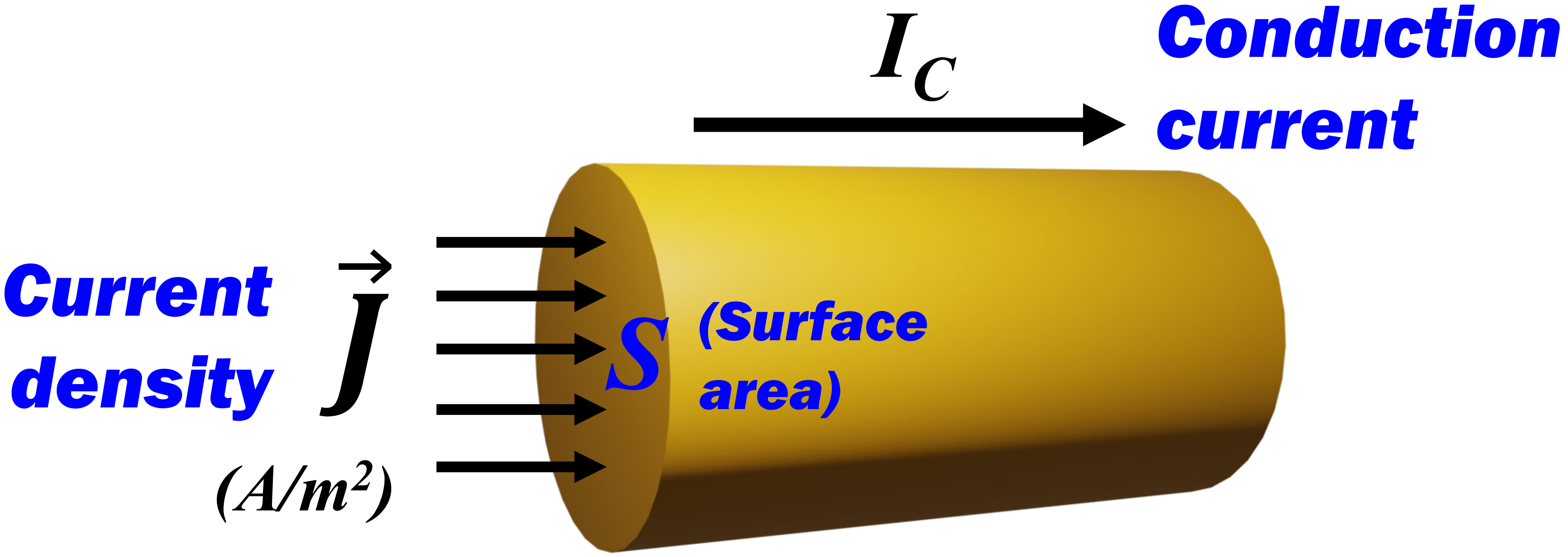
…and If magnetic fields can move conductors, they can certainly couple into circuits and cause disruptions…

Current, Magnetic Field, and Magnetic Flux Density

Magnetic field intensity Ampères/meter $\vec{H}$

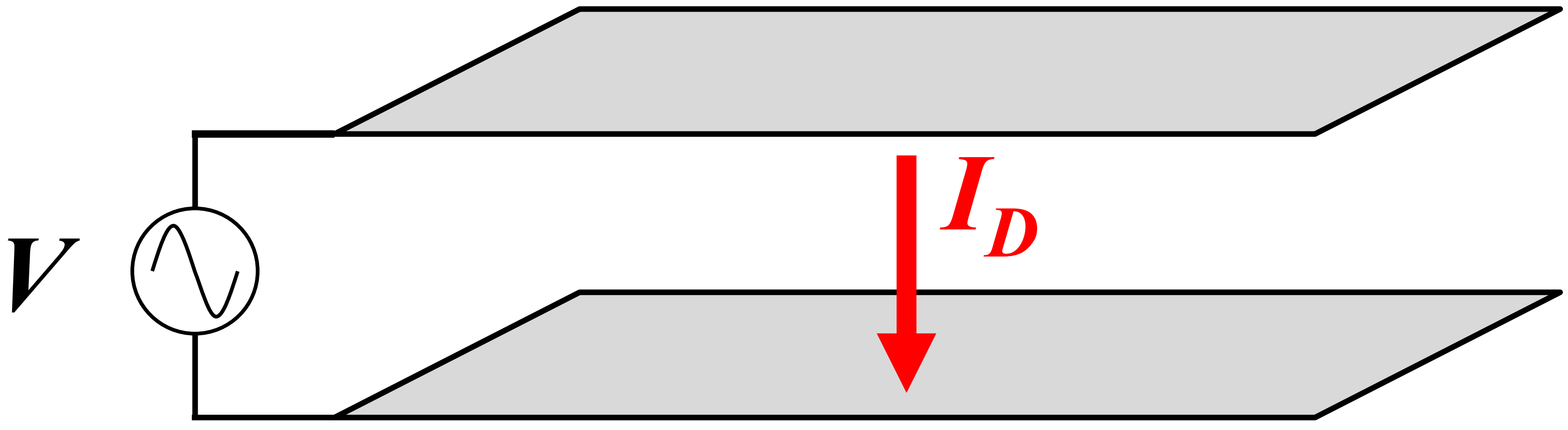
Magnetic flux density Webers/m²
= Tesla $\vec{B}$

Conduction Current



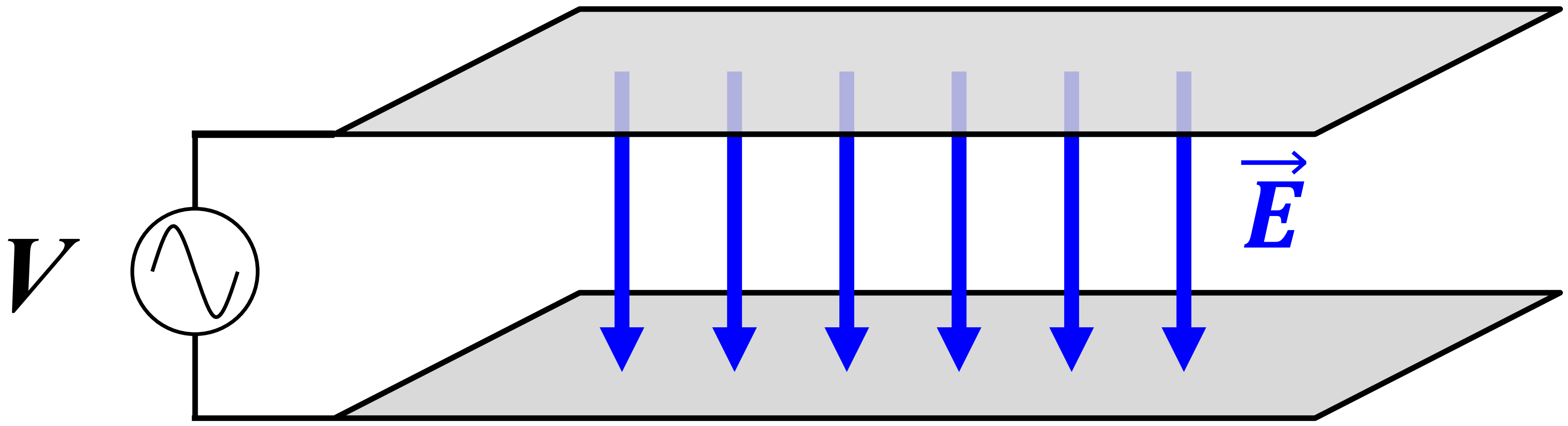
$$I_C = \oiint \vec{J} \cdot d\vec{s}$$

Displacement Current



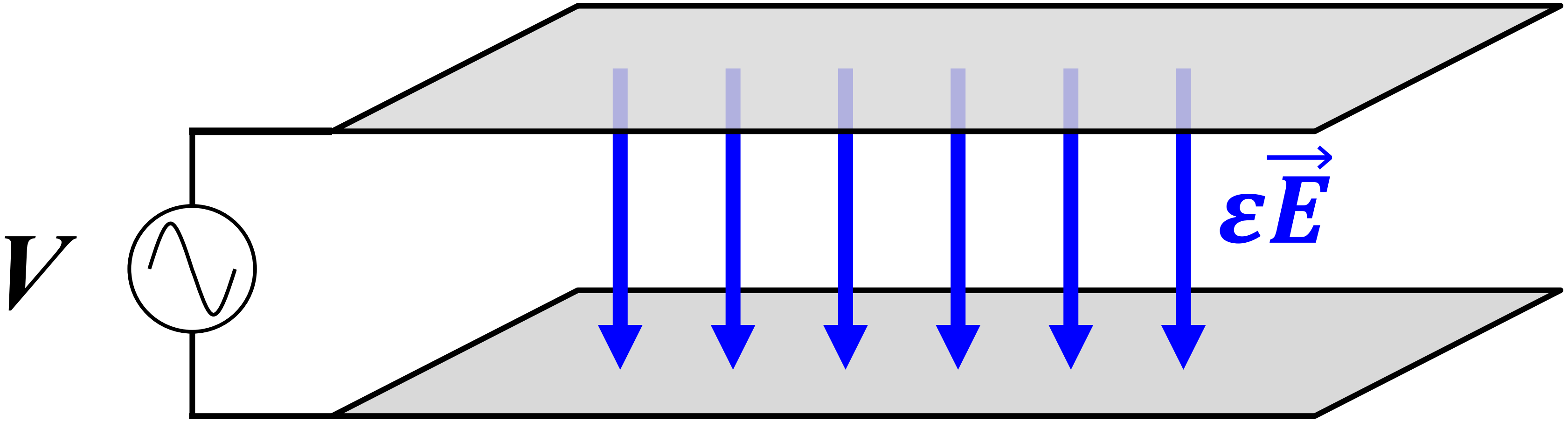
Displacement current (in dielectric)

Displacement Current (cont.)



Displacement current caused by time-varying potential and electric field through dielectric

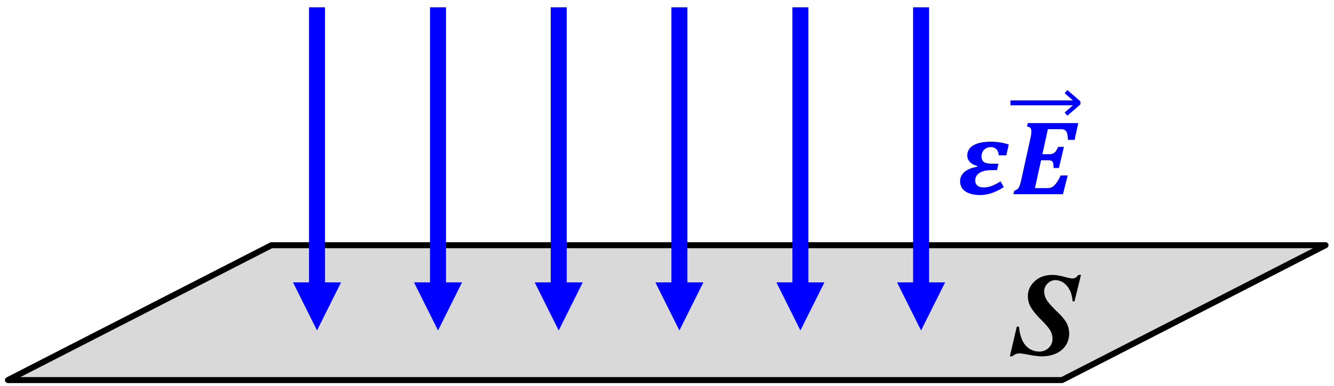
Displacement Current (cont.)



Electric flux density

$$\vec{D} = \varepsilon\vec{E}$$

Displacement Current (cont.)



Total charge ("electric flux") through surface S

$$Q = \oiint \varepsilon \vec{\boldsymbol{E}} \cdot \boldsymbol{d}\vec{\boldsymbol{s}}$$

Displacement current = dQ/dt

$$I_D = \frac{d}{dt} \oiint \varepsilon \vec{\boldsymbol{E}} \cdot \boldsymbol{d}\vec{\boldsymbol{s}}$$

Total Current and Ampère's Law

$$I = I_C + I_D$$

$$I = \underbrace{\oiint \vec{J} \cdot \boldsymbol{d\vec{s}} + \frac{d}{dt} \oiint \varepsilon \vec{\boldsymbol{E}} \cdot \boldsymbol{d\vec{s}}}_{\text{“Right side” of Ampère's Law}}$$

$$\underbrace{\oint \vec{\boldsymbol{H}} \cdot \boldsymbol{d\vec{l}}}_{\text{“Left side” of Ampère's Law}} = I$$

Total Current and Ampère's Law (cont.)

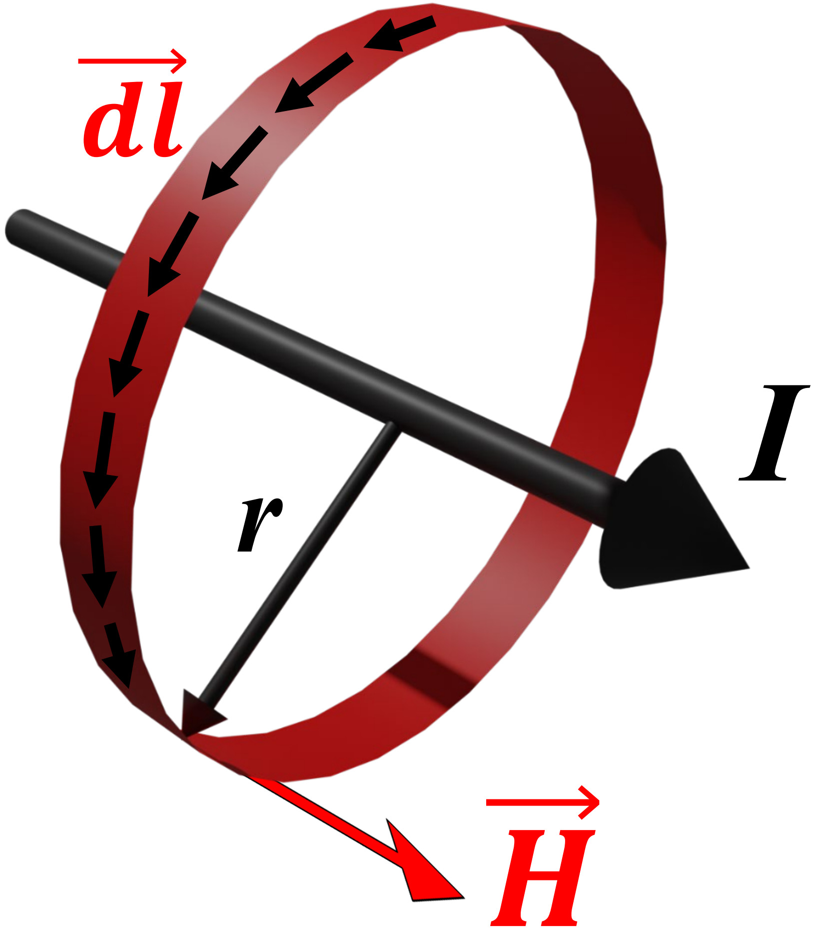
Magnetic field intensity integrated around closed contour equals current passing through surface bounded by contour

$$\oint \vec{H} \cdot d\vec{l} = I$$

For long, straight current-carrying wire:

$$H \cdot 2\pi r = I$$

$$H = \frac{I}{2\pi r}$$



Example:

$I = 1\ A$

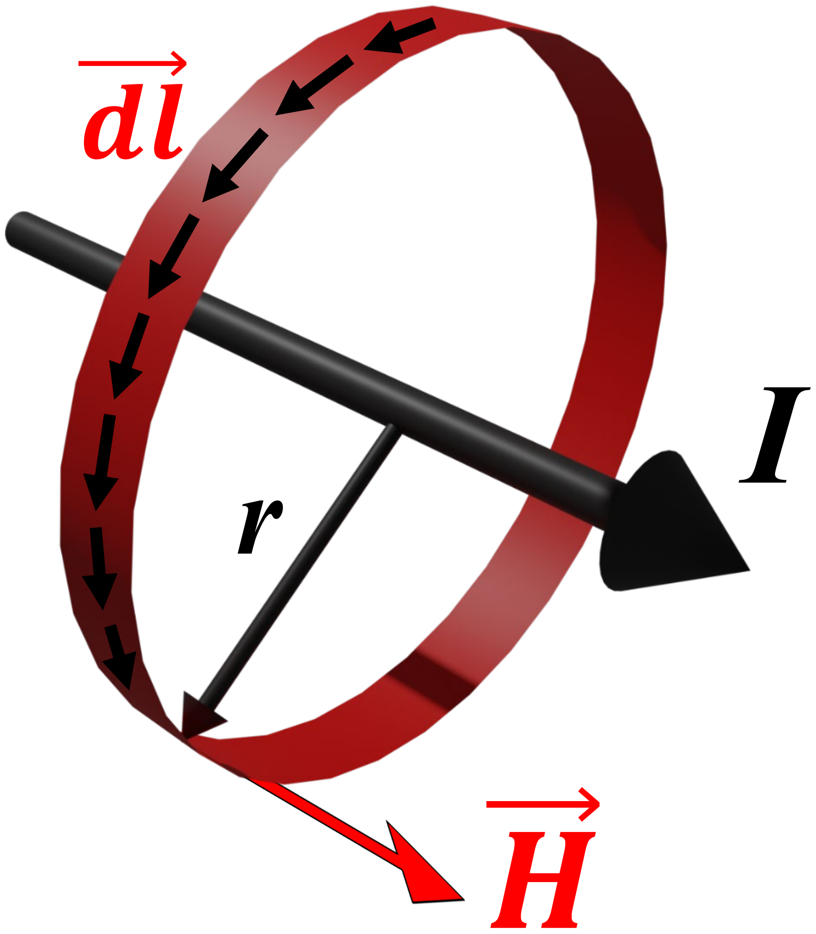
$2\pi r = 1\ m$

$H = 1\ A/m$

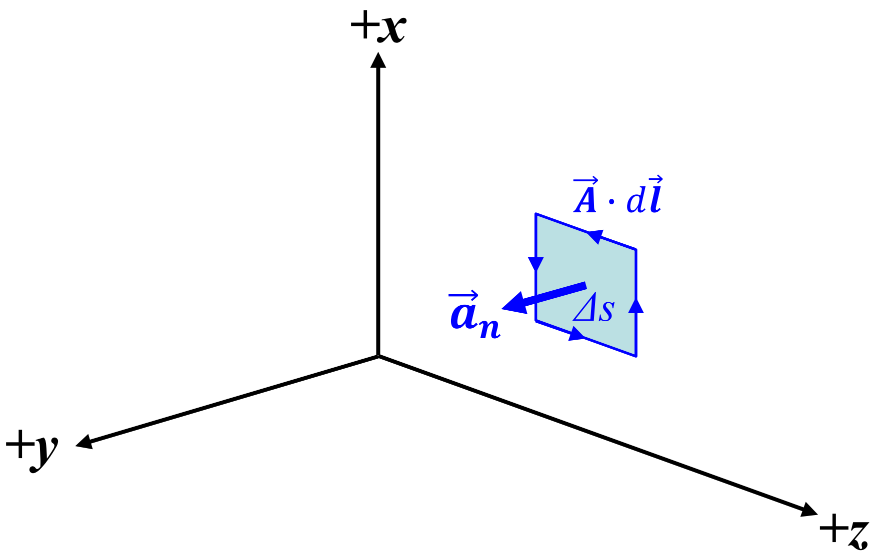
Integral form:

$$\oint \vec{H} \cdot d\vec{l} = \oiint \vec{J} \cdot d\vec{s} + \frac{d}{dt} \oiint \varepsilon \vec{E} \cdot d\vec{s}$$

Differential form…?



Curl of a Vector Field



Definition of "curl":

$$\nabla \times \vec{A} = \lim_{\Delta s \to 0} \frac{1}{\Delta s}\left[\vec{a}_n \oint \vec{A}\cdot d\vec{l}\right]$$

Tendency of vector field to curl or rotate about a point in space

For fellow mathochists:

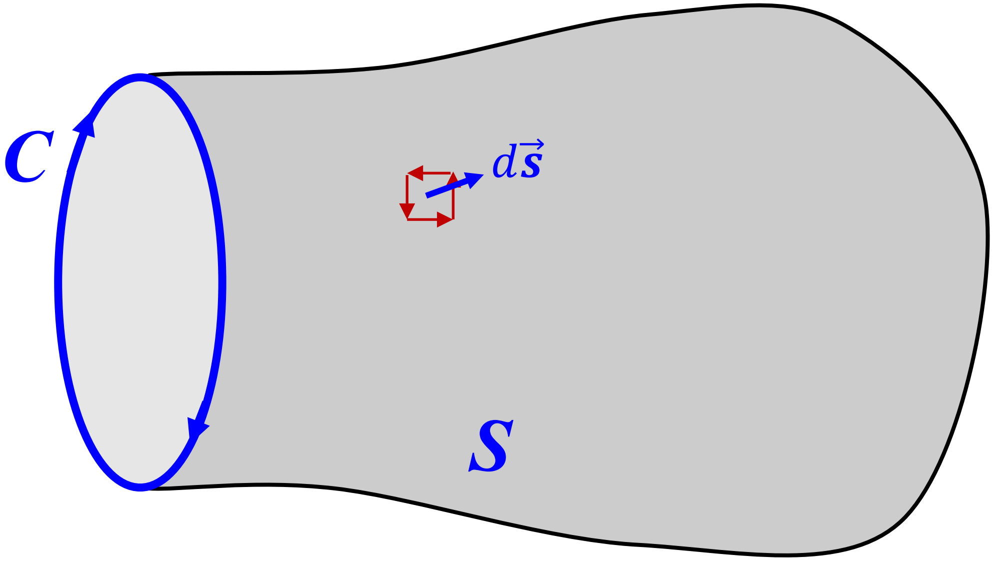
$$\nabla \times \vec{A} = \begin{bmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{bmatrix}$$

$$= \vec{a}_x\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \vec{a}_y\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \vec{a}_z\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

Stokes' Theorem

$$\oiint \left(\nabla \times \vec{A}\right) \cdot d\vec{s} = \oint \vec{A} \cdot d\vec{l}$$

The surface integral of the curl of a vector field over an open surface equals the closed line integral of the vector along the contour bounding the surface.



Applying to Ampère's Law:

$$\oint \vec{H} \cdot d\vec{l} = \oiint \vec{J} \cdot d\vec{s} + \frac{d}{dt} \oiint \varepsilon \vec{E} \cdot d\vec{s}$$

$$\oiint \left(\nabla \times \vec{H}\right) \cdot d\vec{s} = \oiint \vec{J} \cdot d\vec{s} + \frac{d}{dt} \oiint \varepsilon \vec{E} \cdot d\vec{s}$$

All surface integrals

Integrands are equal

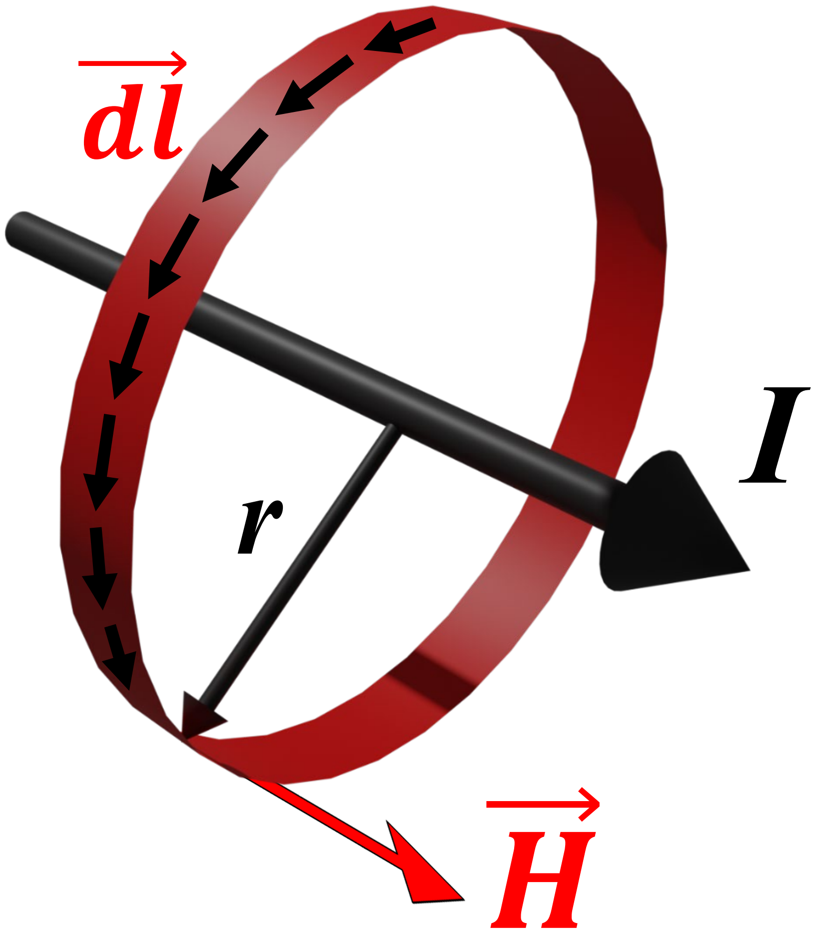
$$\nabla \times \vec{H} = \vec{J} + \varepsilon \frac{d\vec{E}}{dt}$$

Integral form:

$$\oint \vec{H} \cdot d\vec{l} = \oiint \vec{J} \cdot d\vec{s} + \frac{d}{dt} \oiint \varepsilon \vec{E} \cdot d\vec{s}$$

Differential form:

$$\nabla \times \vec{H} = \vec{J} + \varepsilon \frac{d\vec{E}}{dt}$$



Right Hand Rule

Point thumb of right hand in direction of current flow

Fingers wrap in direction of magnetic field

Permeability and Permittivity

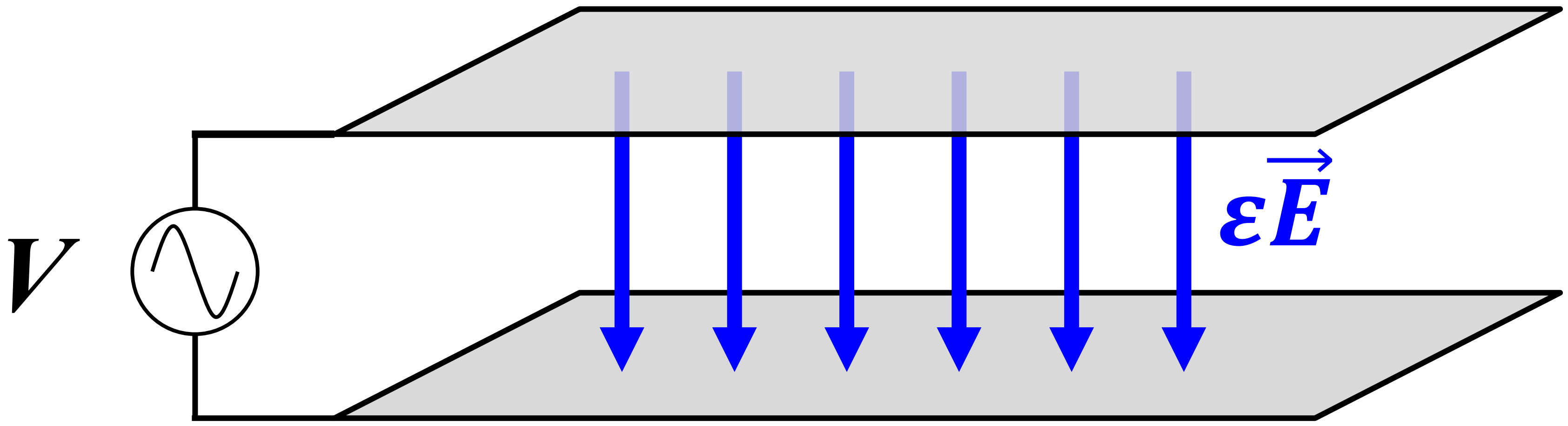
$$\vec{B} = \mu \vec{H}$$

Permeability

$$\vec{D} = \varepsilon \vec{E}$$

Permittivity

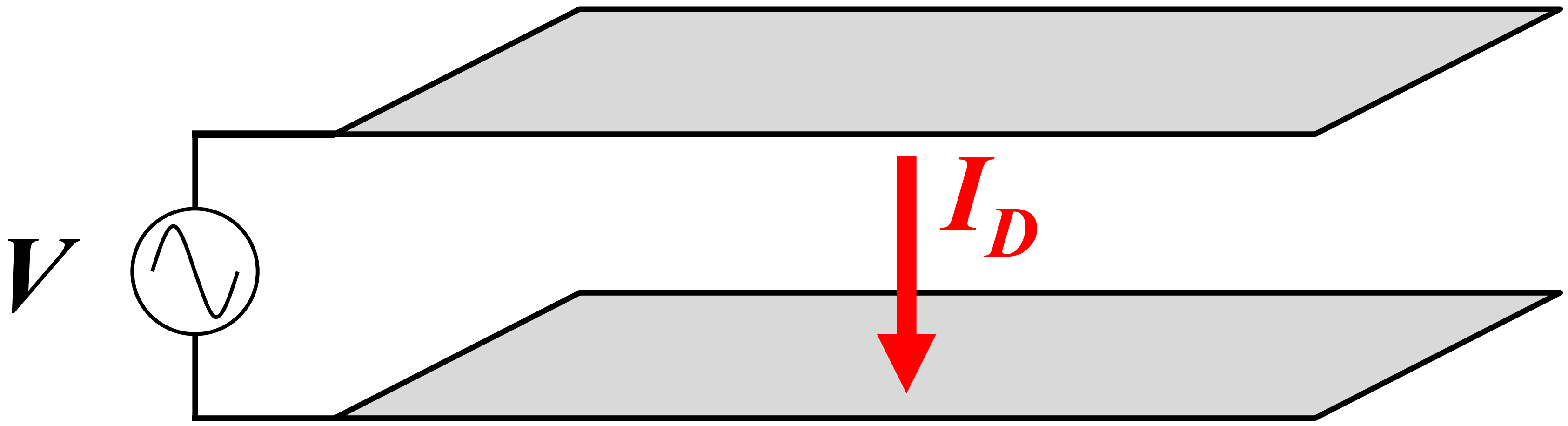
Recall: Permittivity and Displacement Current



Permittivity determines available charge…

$$Q = \oint\!\!\!\oint \varepsilon \vec{E} \cdot d\vec{s}$$

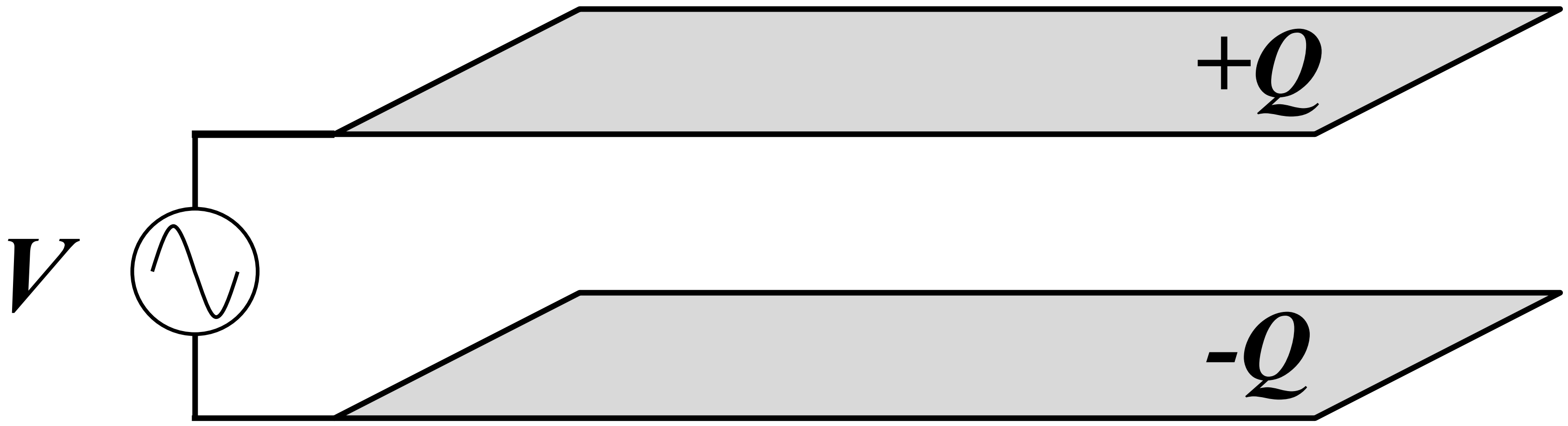
Permittivity and Displacement Current (cont.)



...that can form displacement current in electric field

$$I_D = \frac{d}{dt} \oiint \varepsilon \vec{\boldsymbol{E}} \cdot \boldsymbol{d}\vec{\boldsymbol{s}}$$

Permittivity and Displacement Current (cont.)



On larger scale, capacitance determines available charge for given potential

$$C = \frac{Q}{V}$$

$$\vec{\boldsymbol{B}} = \mu \vec{\boldsymbol{H}}$$

Permeability determines available magnetic flux from given magnetic field that can cause electromagnetic induction

Permeability (cont.)

$$\mu = \mu_r \mu_0$$

$$\mu_r = relative\ permeability$$

$$\mu_0 = permeability\ of\ free\ space$$

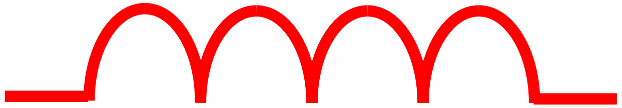
Permeability (cont.)

$$\mu = \mu_r \mu_0$$

$$\mu_0 = permeability\ of\ free\ space$$

$$\mu_0 = 4\pi \times 10^{-7}\,\frac{Henries}{m}$$

Henries (H) are units of inductance



$$\mu_0 \approx 1.26\ \mu H/m$$

“Inductivity”

As with resistivity and permittivity, “per meter” takes on physical meaning only when applied to specific geometry

Permeability (cont.)

$$\mu = \mu_r \mu_0$$

$$\mu_r = relative\ permeability$$

Ferromagnetic materials:

$$\mu_r\,(nickel) \approx 250$$

$$\mu_r\,(cobalt) \approx 600$$

$$\mu_r\,(iron) \approx 4000$$

Non-ferromagnetic:

Air ***Aluminum***

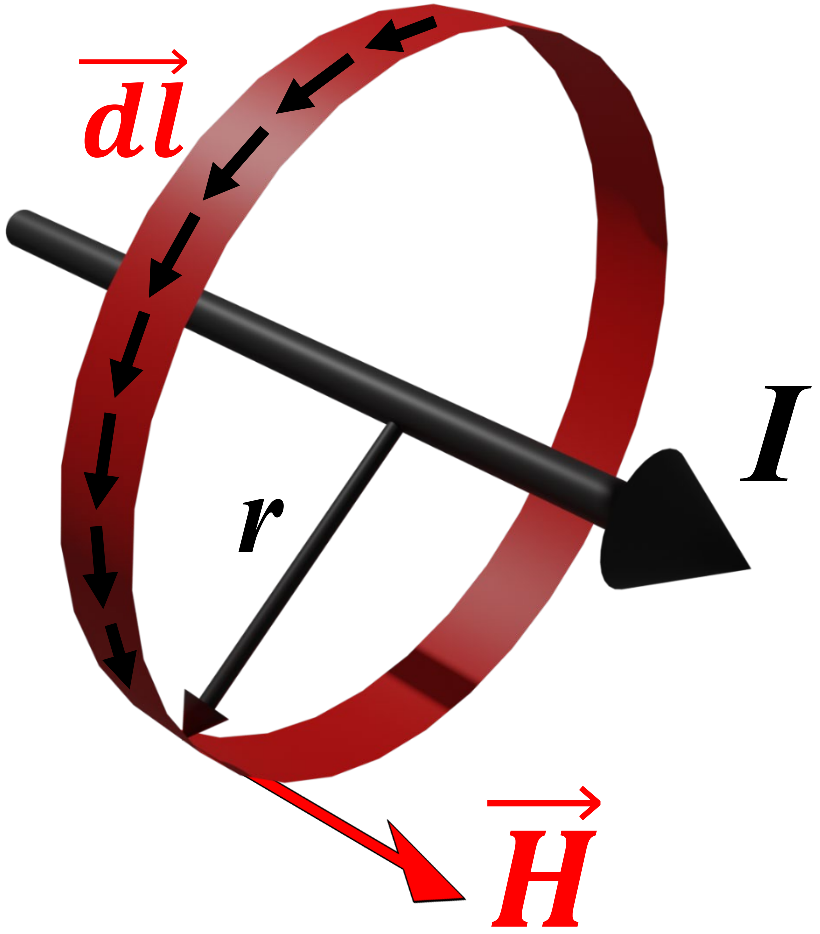
Wire insulation ***Copper***

etc.

$$\mu_r \approx 1$$

Magnetic Field Intensity and Magnetic Flux Density

$$\oint \vec{H} \cdot d\vec{l} = I$$



Magnetic Field Intensity and Magnetic Flux Density (cont.)

$$\oint \vec{H} \cdot d\vec{l} = I$$

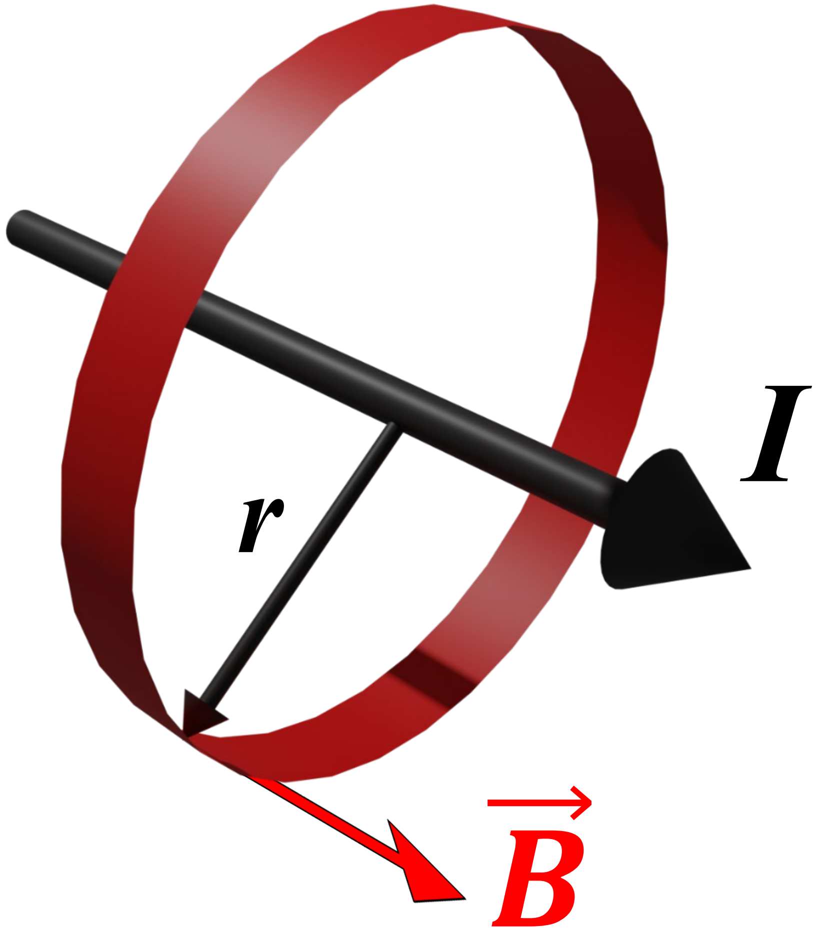
Magnetic flux density:

$$\vec{B} = \mu\vec{H}$$

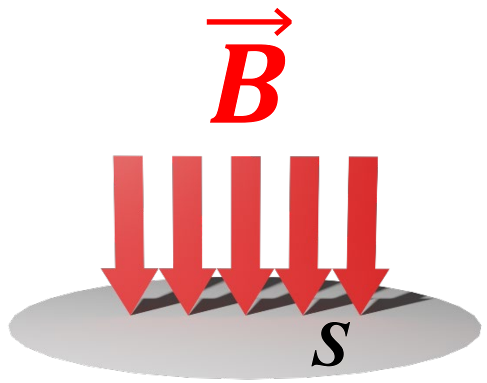
Webers/m^2 = Tesla (T)

1 T = 10^4 gauss

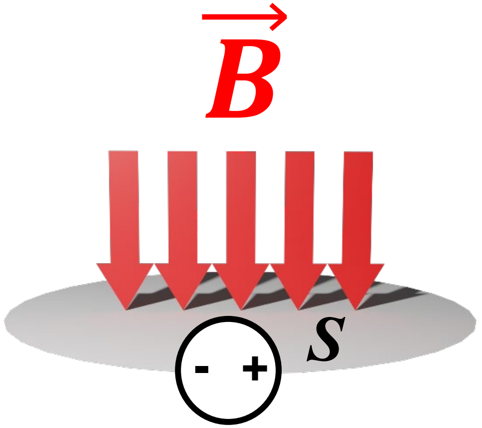
1 gauss = 0.1 mT



Total Magnetic Flux



$$\Phi = \oiint \vec{B} \cdot d\vec{s}$$

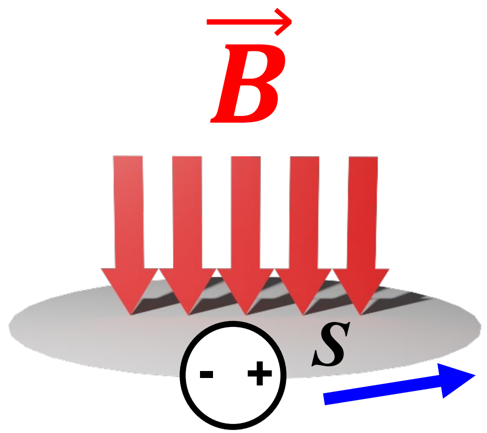


Time-varying magnetic flux through surface induces potential (electromotive force, a.k.a. "emf") in contour surrounding loop

$$V_{emf} = -\frac{d\Phi}{dt}$$

$$= -\frac{d}{dt}\oiint \vec{\boldsymbol{B}} \cdot \boldsymbol{d}\vec{\boldsymbol{s}}$$

Induced Potential and Lenz's Law



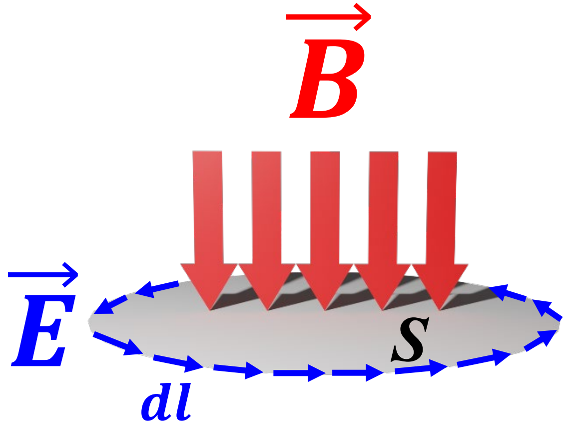
$$V_{emf} = -\frac{d\Phi}{dt}$$

Induced emf opposes (impedes) incident flux

Lenz's Law

Faraday’s Law of Electromagnetic Induction

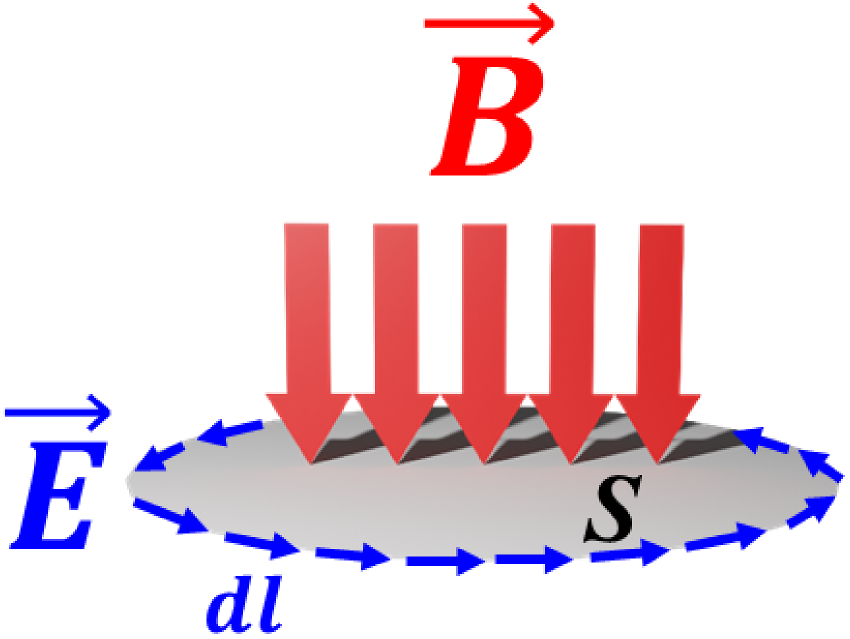
$$V_{emf} = \oint \vec{\boldsymbol{E}} \cdot \boldsymbol{d}\vec{\boldsymbol{l}}$$



$$\oint \vec{\boldsymbol{E}} \cdot \boldsymbol{d}\vec{\boldsymbol{l}} = -\frac{d}{dt}\oiint \vec{\boldsymbol{B}} \cdot \boldsymbol{d}\vec{\boldsymbol{s}}$$

Faraday’s Law of Electromagnetic Induction

Faraday's Law of Electromagnetic Induction (cont.)



Integral form:

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \oiint \vec{B} \cdot d\vec{s}$$

$\Big\downarrow$ Stokes' Theorem

$$\oiint (\nabla \times \vec{E}) \cdot d\vec{s} = -\frac{d}{dt} \oiint \vec{B} \cdot d\vec{s}$$

Differential form:

$$\nabla \times \vec{E} = -\frac{d\vec{B}}{dt}$$

Faraday's Law of Electromagnetic Induction (cont.)

$$\oint \vec{\boldsymbol{E}} \cdot \boldsymbol{d}\vec{\boldsymbol{l}} = -\frac{d}{dt} \oiint \vec{\boldsymbol{B}} \cdot \boldsymbol{d}\vec{\boldsymbol{s}}$$

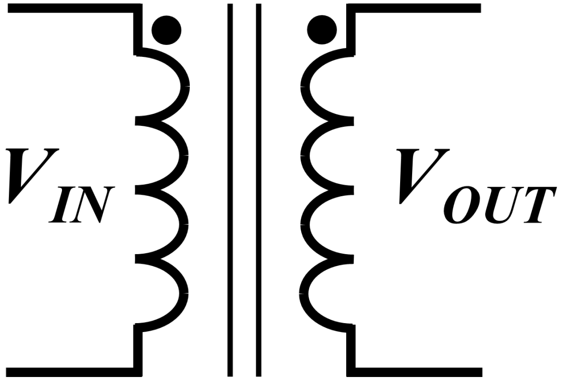
It is impossible to overstate the significance of Faraday's Law

$$V_{emf} = -\frac{d\Phi}{dt}$$

It makes many things possible…

Faraday's Law: The World Would be Awfully Dull Without It…

Power generation (conversion of mechanical power to electrical power)



Power distribution over large areas

Image courtesy of NASA

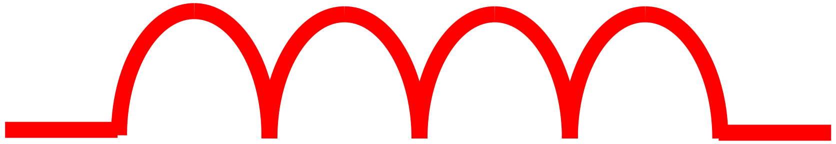
Faraday’s Law and Inductance

Faraday’s Law of Electromagnetic INDUCTION…

$$\oint \vec{\boldsymbol{E}} \cdot \boldsymbol{d}\vec{\boldsymbol{l}} = -\frac{d}{dt}\oiint \vec{\boldsymbol{B}} \cdot \boldsymbol{d}\vec{\boldsymbol{s}}$$

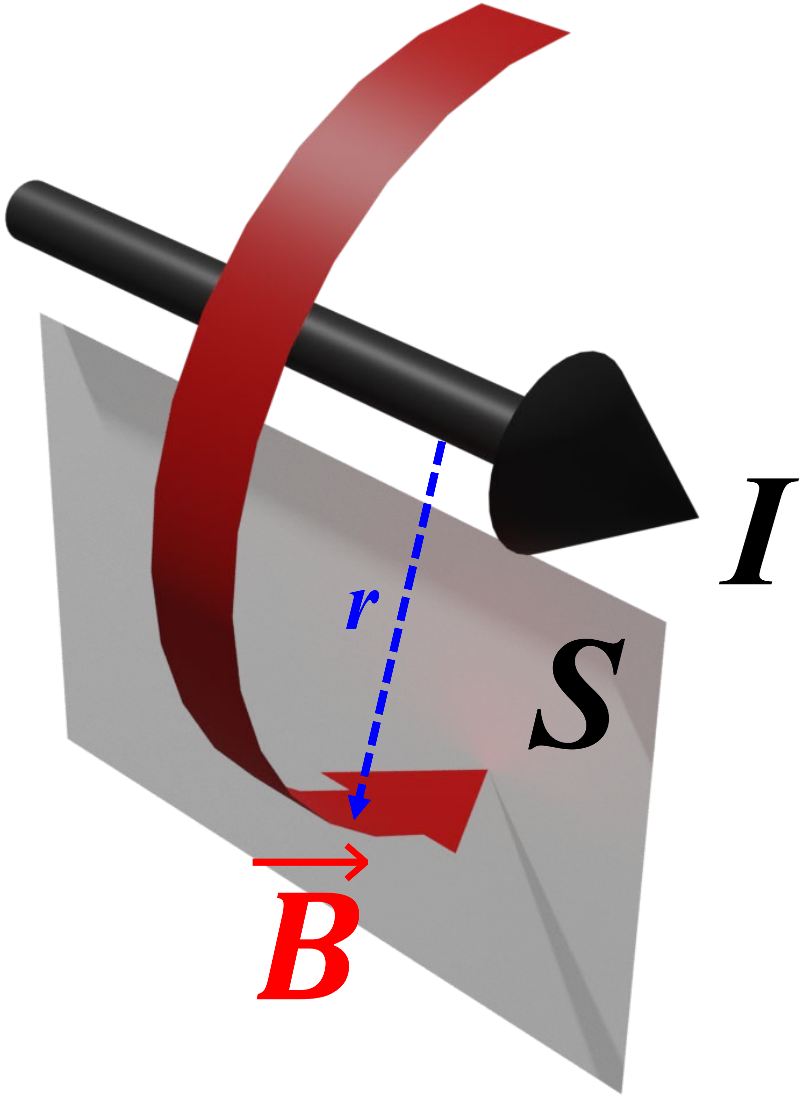
$$V_{emf} = -\frac{d\Phi}{dt}$$

INDUCED potential…



INDUCTANCE

Inductance Definition



$$B = \frac{\mu I}{2\pi r}$$

$$\Phi = \oiint \vec{B} \cdot d\vec{s}$$

$$\mathrm{L} = \frac{\Phi}{\mathrm{I}}$$

Magnetic flux per unit current

Self Inductance and Mutual Inductance

Self inductance
(same loop)

Mutual inductance
(neighboring loop)

Inductance Units

$$B = \frac{\mu I}{2\pi r} \qquad \frac{(H/m)\cdot A}{m}$$

$$\Phi = \oiint \vec{\boldsymbol{B}} \cdot \boldsymbol{d}\vec{\boldsymbol{s}} \qquad \frac{H \cdot A}{m^2} \cdot m^2$$

$$\mathrm{L} = \frac{\Phi}{\mathrm{I}} \qquad \frac{H}{m^2} \cdot m^2$$

$$L(units) = H$$

Inductive Reactance (Impedance)

$$V_{emf} = -\frac{d\Phi}{dt}$$

$$|V| = \frac{d\Phi}{dt} \qquad \mathrm{L} = \frac{\Phi}{\mathrm{I}} \qquad \Phi = \mathrm{LI}$$

$$I(t) = Ie^{j\omega t}$$

$$|V| = L\frac{dI}{dt} \qquad \frac{dI(t)}{dt} = j\omega I e^{j\omega t}$$

$$|V| = j\omega LI$$

Inductive reactance

$$\frac{|V|}{I} = X_L = j\omega L = j2\pi f L$$

Increases with frequency

Self inductance:

For given potential, resulting current decreases with frequency

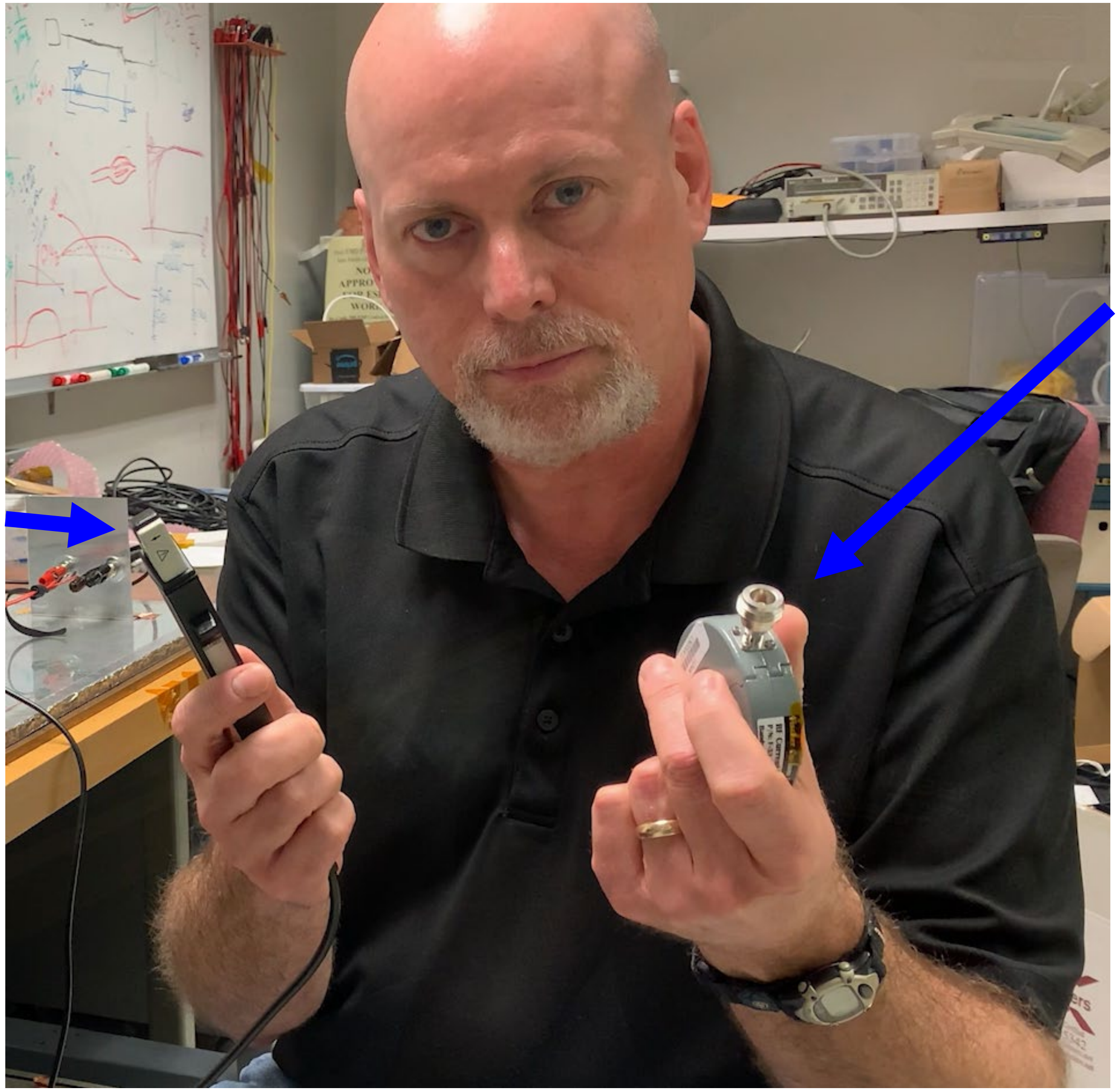
Mutual inductance:

Coupling increases with frequency

Practical Applications: Current Probes

Hall Effect current probe

Used for time domain measurements down to DC; generally made for specific oscilloscopes



Clamp-on magnetic core current probe

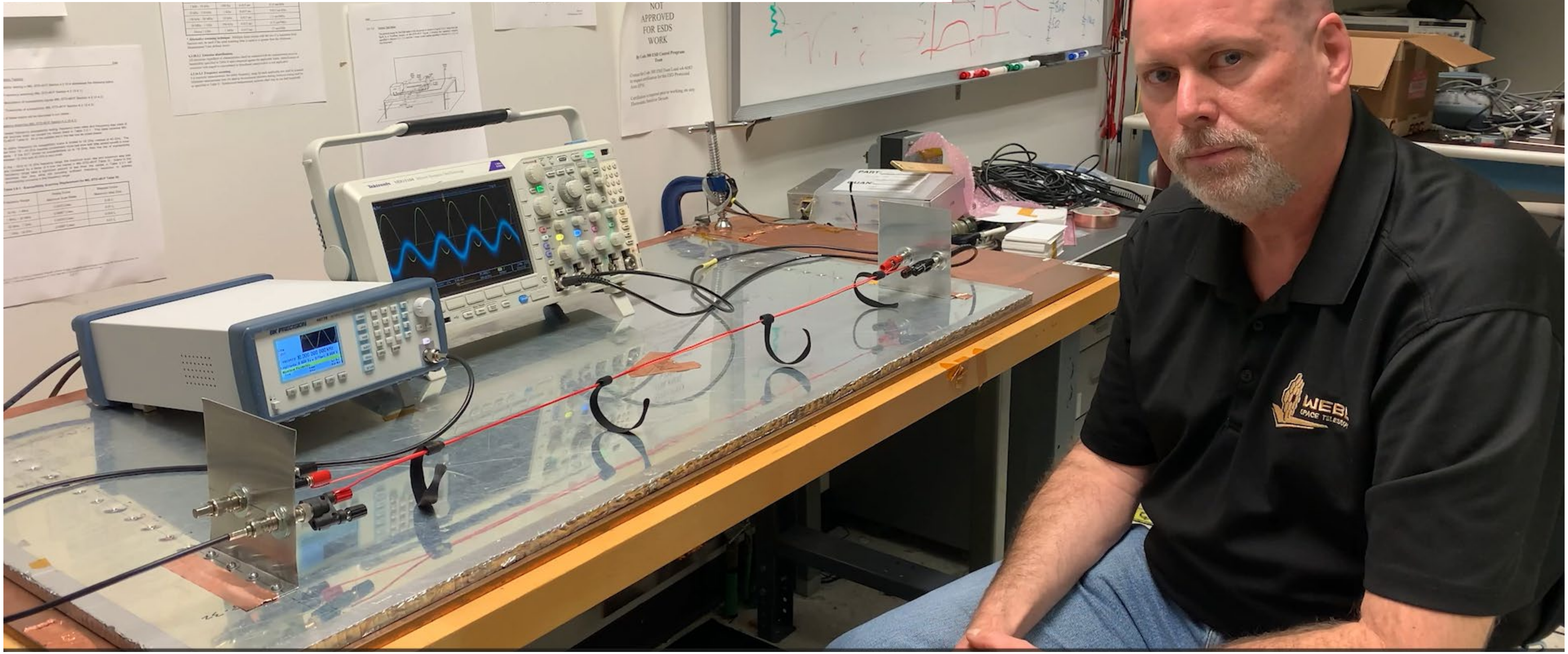
Clamps on wire/cable of interest with no disconnections

Captures magnetic flux generated by current on wire/cable of interest (Ampère's Law)

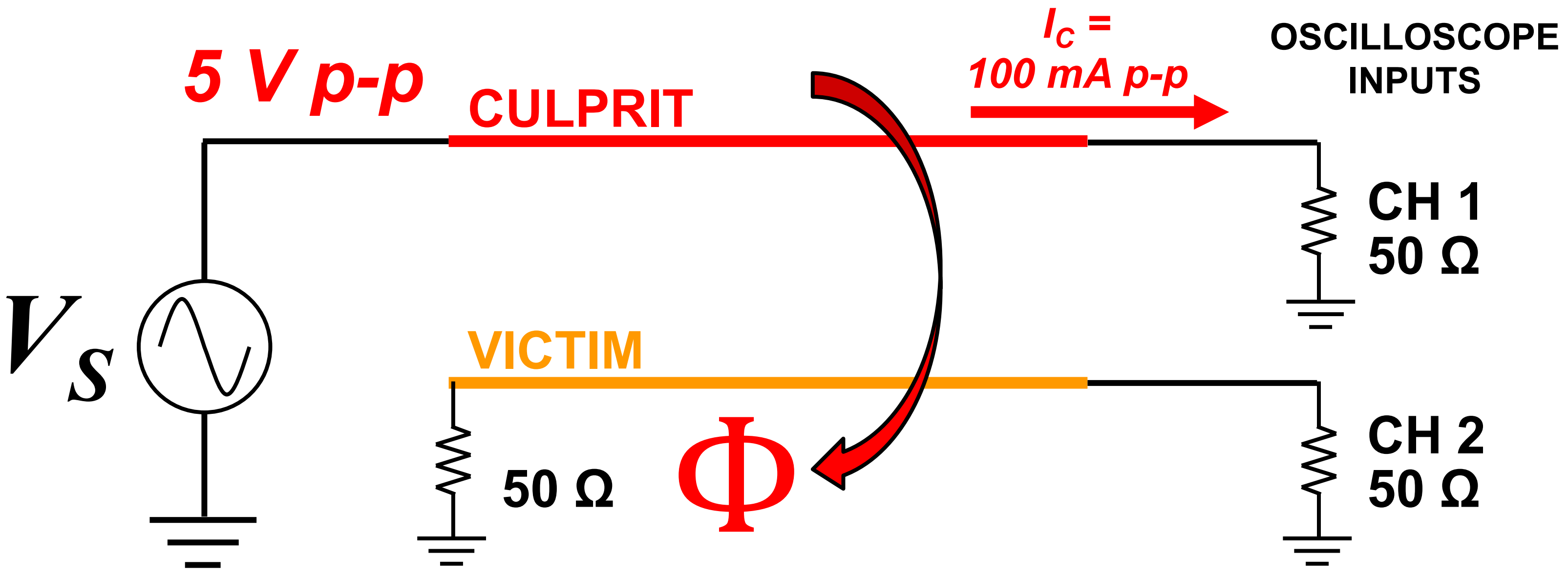
Induces potential in coil of wire wrapped around core connected to output connector (Faraday's Law)

(Virtual) Demonstration: Inductive Coupling

Details in "Fundamentals of Electromagnetics" video: Magnetic Field, Current, and Inductance – Part 1



(Virtual) Demonstration: Inductive Coupling (cont.)



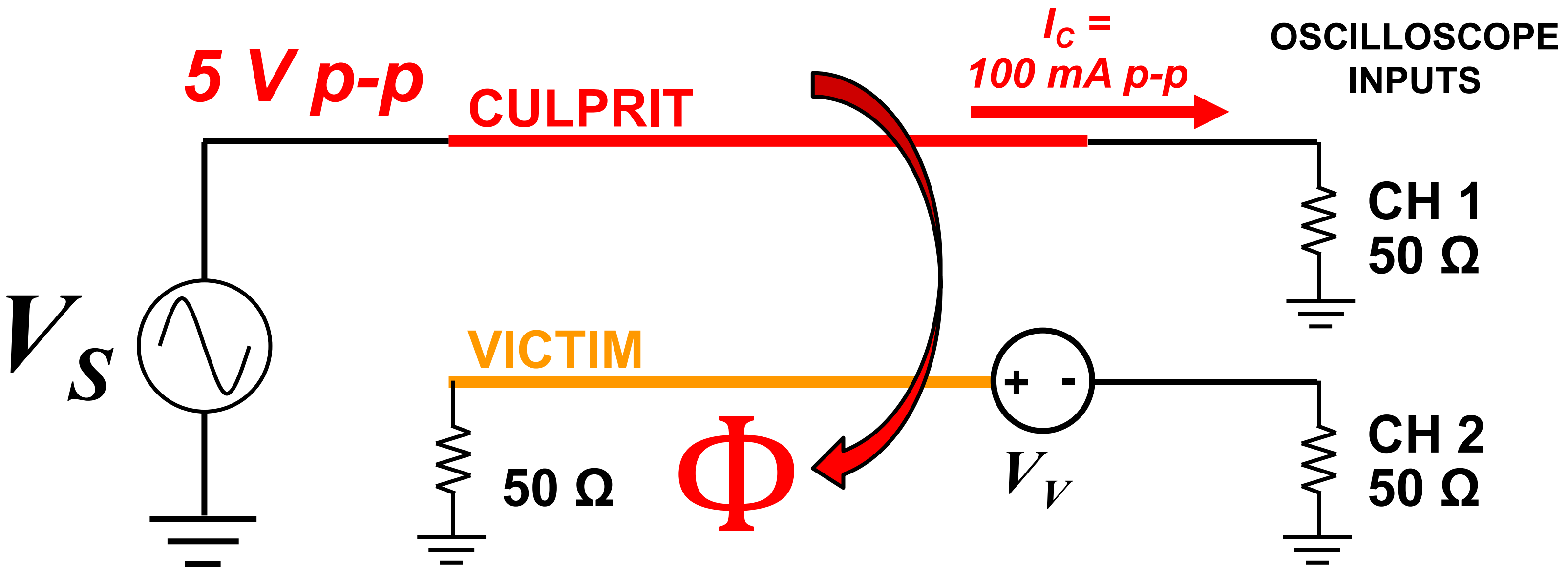
Magnetic flux points INTO page (right hand rule)

$$\Phi = L_M I_C$$

$$L_M \approx 700\ nH$$

Derived in backup slides and in Fundamentals video:
"Magnetic Field, Current, and Inductance – Part 2"

(Virtual) Demonstration: Inductive Coupling (cont.)

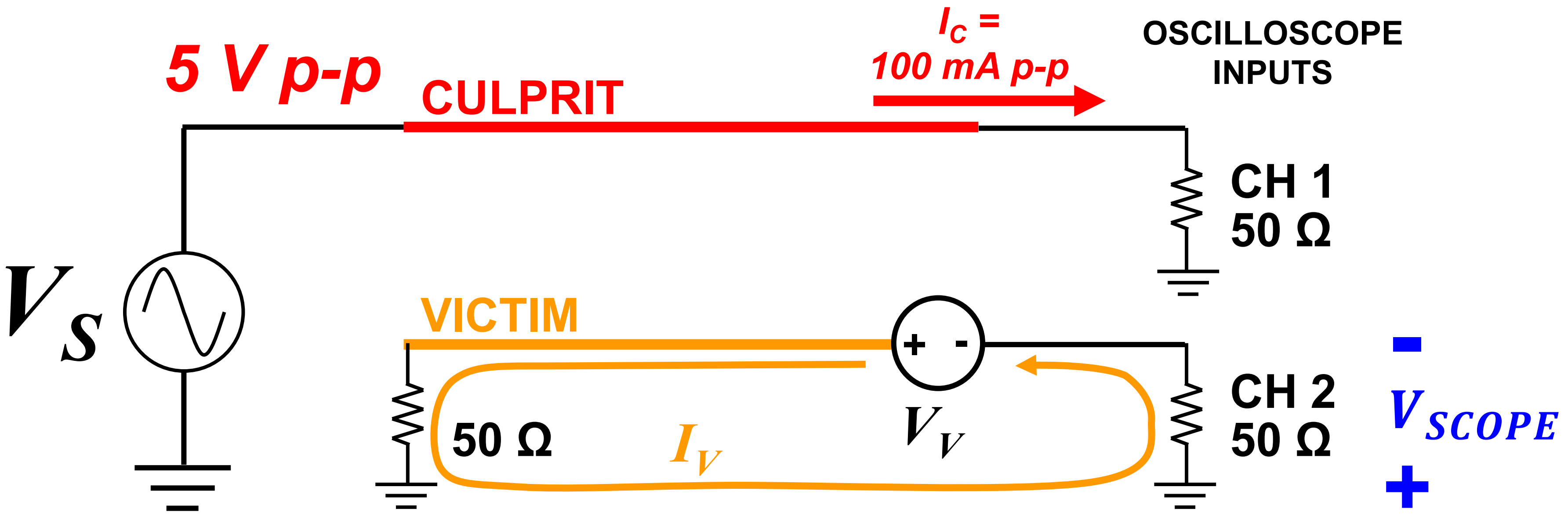


$$V_V = \frac{d\Phi}{dt} = L_M \frac{dI_C}{dt}$$

$$V_V = j\omega L_M I_C = j2\pi f L_M I_C$$

Induced potential opposes (impedes) incident flux (Lenz's Law)

(Virtual) Demonstration: Inductive Coupling (cont.)

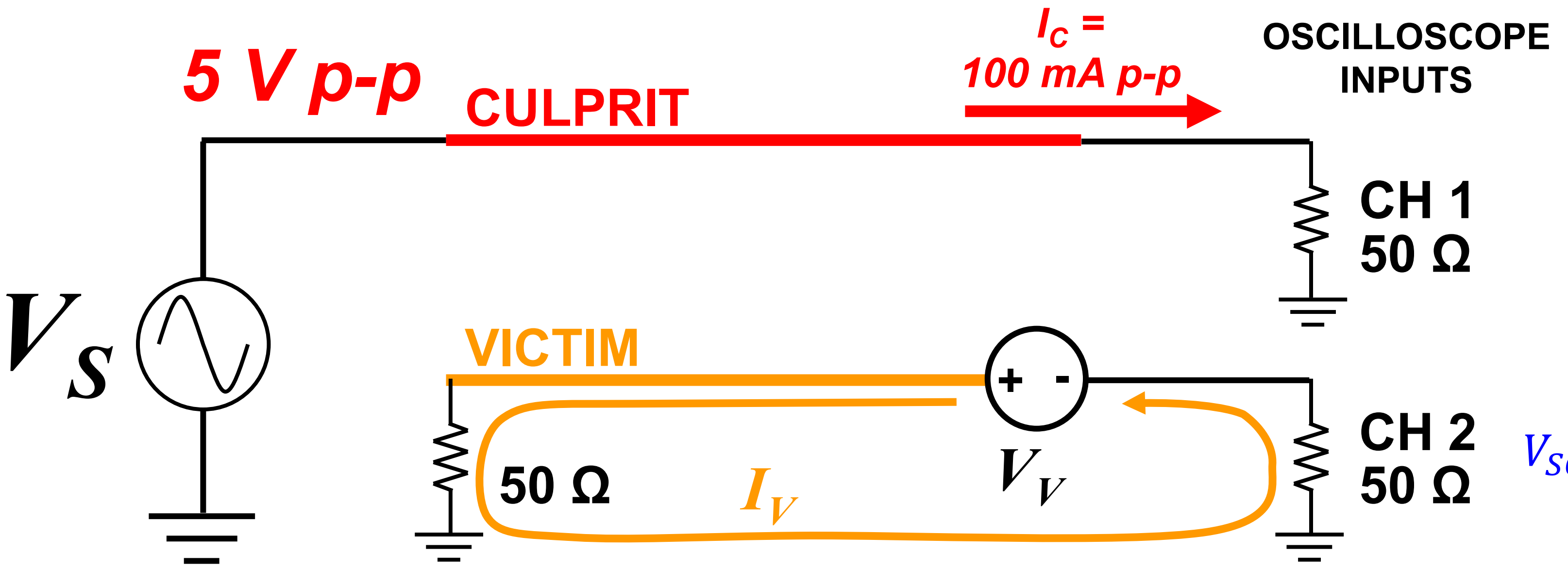


$$V_{SCOPE} = -\frac{V_V}{2}$$

$$V_{SCOPE} = -j\pi f L_M I_C$$

V_{SCOPE} negative peaks will lead I_C by 90º (V leads I by 90º in inductor)

(Virtual) Demonstration: Inductive Coupling (cont.)



$V_{SCOPE} = -j\pi f L_M I_C$

At starting frequency f = 100 kHz:

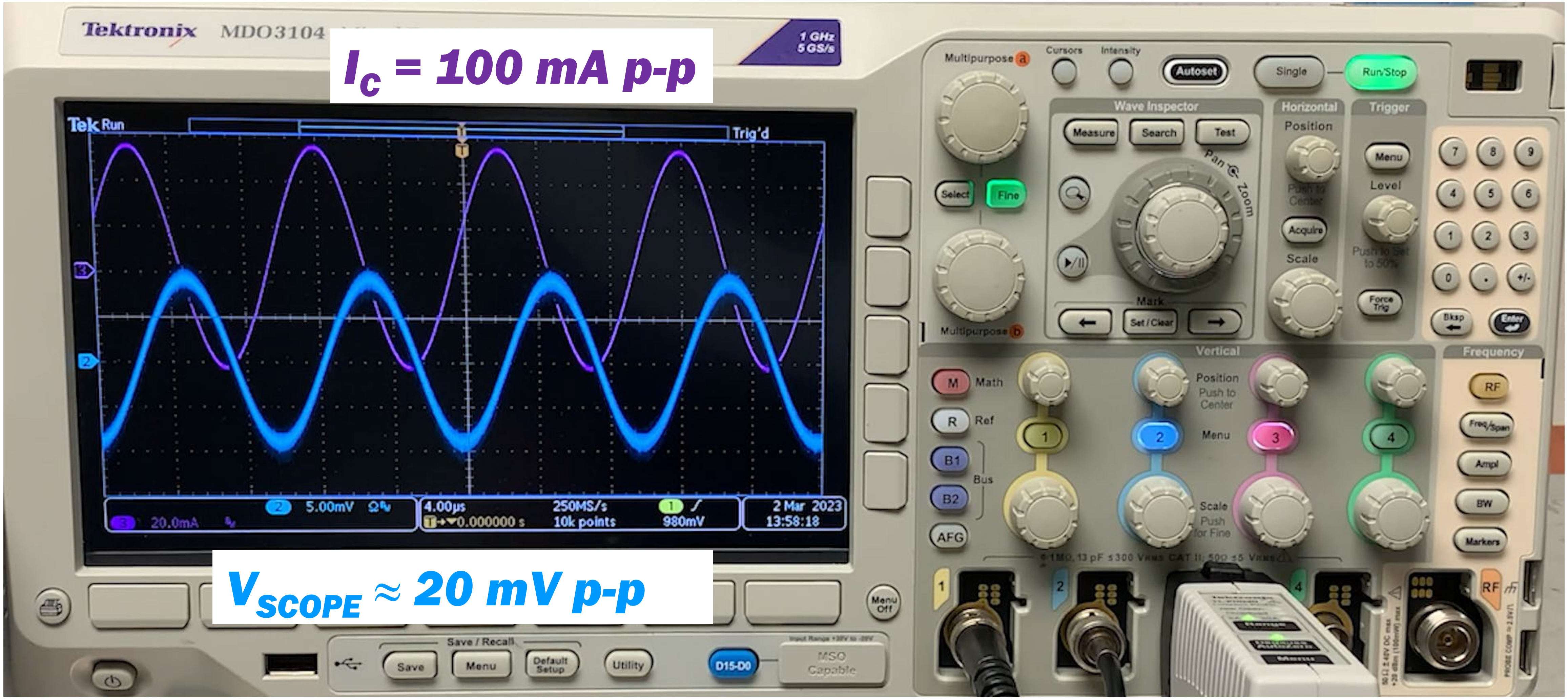
$$|V_{SCOPE}| \approx \pi(100\ kHz)(700\ nH)(100\ mA\ p-p)$$

$$|V_{SCOPE}| \approx 22\ mV\ p-p$$

(Virtual) Demonstration: Inductive Coupling (cont.)

f = 100 kHz

I_C = 100 mA p-p

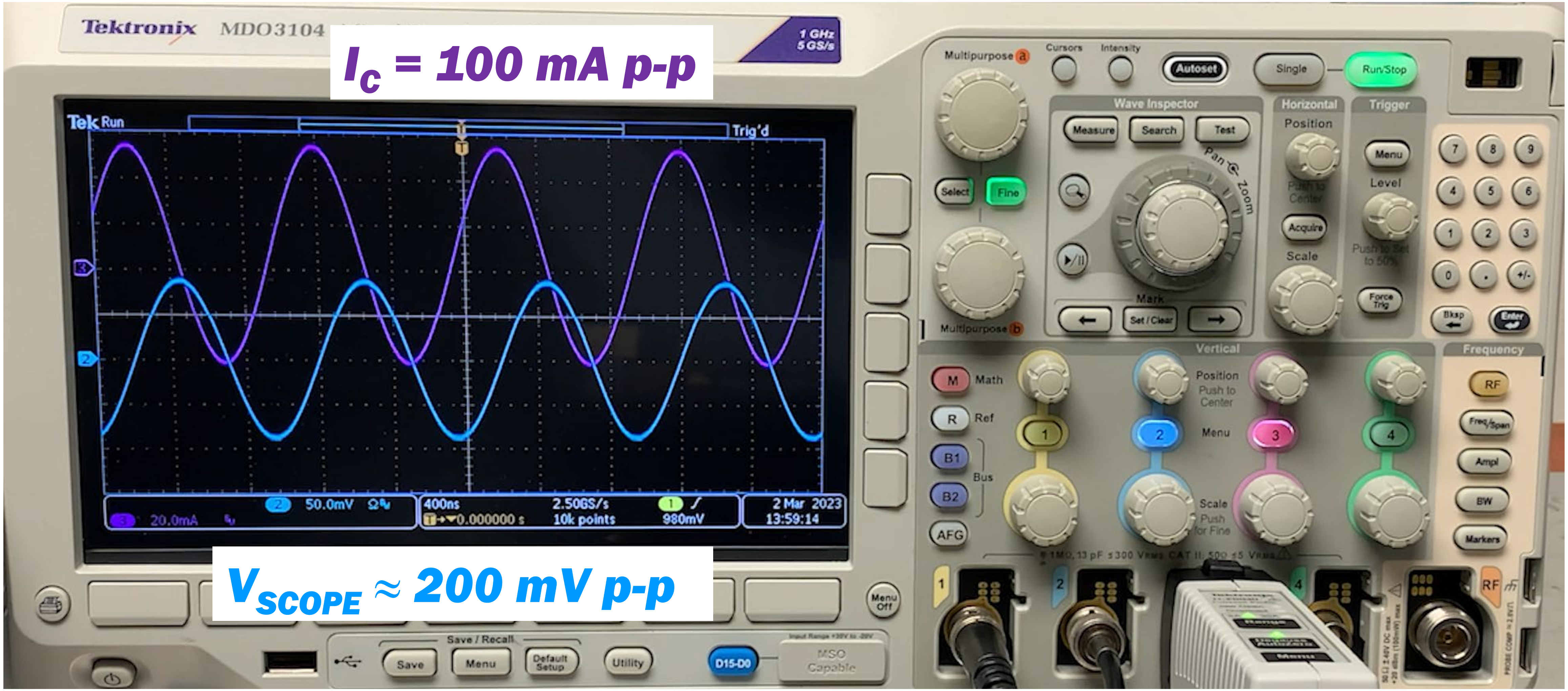


$V_{SCOPE} \approx$ 20 mV p-p

(Virtual) Demonstration: Inductive Coupling (cont.)

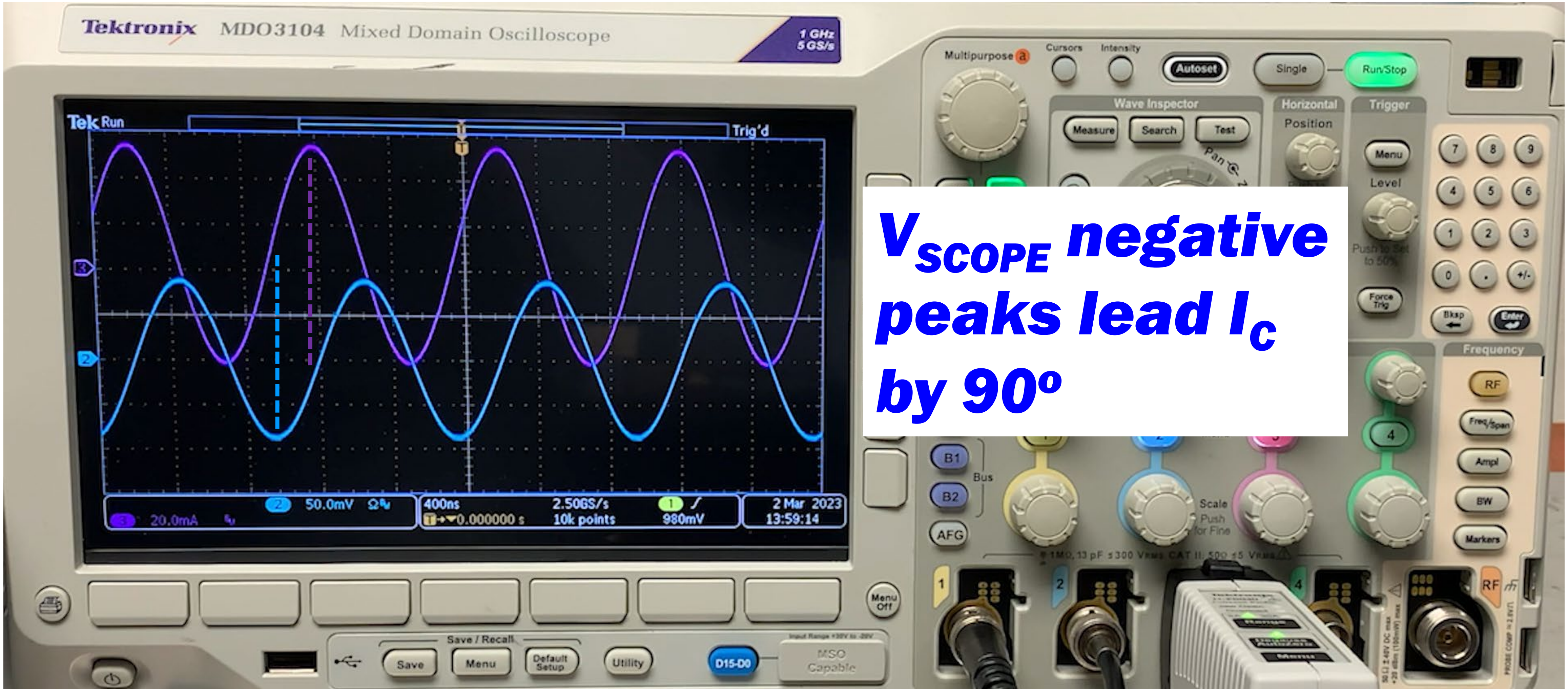
***f* = 1 MHz**

I_C = 100 mA p-p



V_{SCOPE} ≈ 200 mV p-p

(Virtual) Demonstration: Inductive Coupling (cont.)



V_{SCOPE} negative peaks lead I_C by 90º

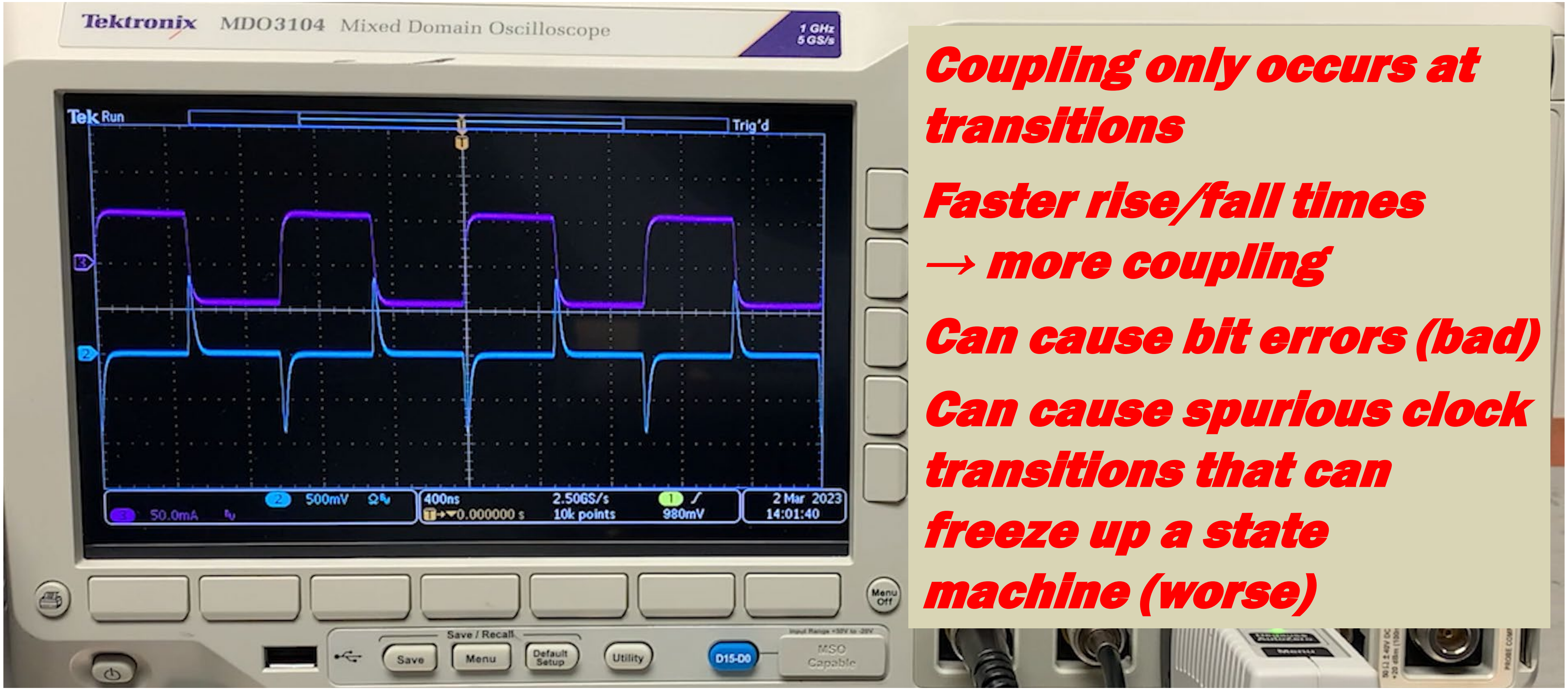
Is this amount of coupling a problem?

It depends…

"But I'm a digital logic designer,
and I only care about 1's and 0's.
What do I care about any of this?"

It's déja vu all over again…

Virtual Demonstration: Inductive Coupling (cont.)



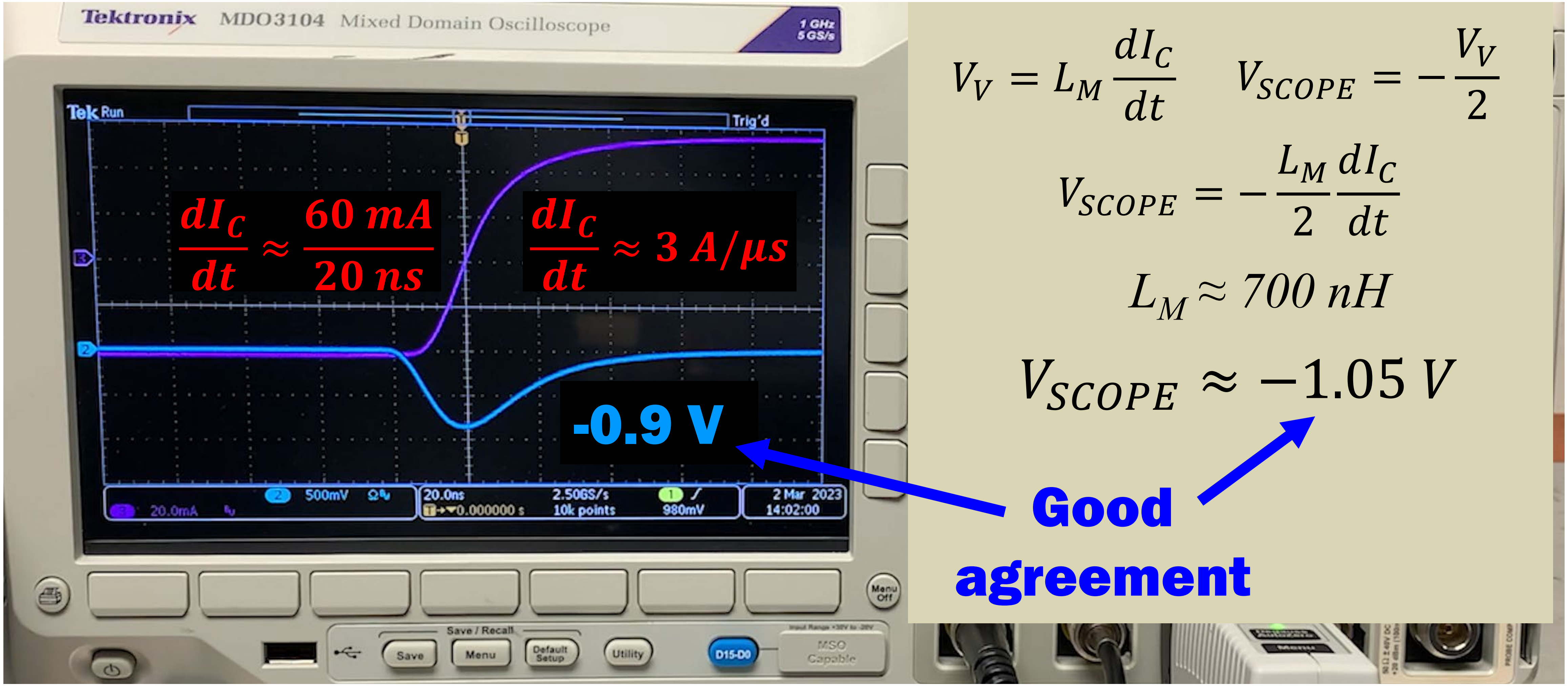
Coupling only occurs at transitions

Faster rise/fall times → more coupling

Can cause bit errors (bad)

Can cause spurious clock transitions that can freeze up a state machine (worse)

Virtual Demonstration: Inductive Coupling (cont.)



$$\frac{dI_C}{dt} \approx \frac{60\ mA}{20\ ns} \qquad \frac{dI_C}{dt} \approx 3\ A/\mu s$$

-0.9 V

$$V_V = L_M \frac{dI_C}{dt} \qquad V_{SCOPE} = -\frac{V_V}{2}$$

$$V_{SCOPE} = -\frac{L_M}{2}\frac{dI_C}{dt}$$

$$L_M \approx 700\ nH$$

$$V_{SCOPE} \approx -1.05\ V$$

Good agreement

Magnetic Field, Current, and Inductance

$$\oint \vec{H} \cdot d\vec{l} = \oiint \vec{J} \cdot d\vec{s} + \frac{d}{dt} \oiint \varepsilon \vec{E} \cdot d\vec{s}$$

$$\vec{B} = \mu \vec{H}$$

Use Ampère's Law to calculate magnetic field and magnetic flux density as function of distance from current

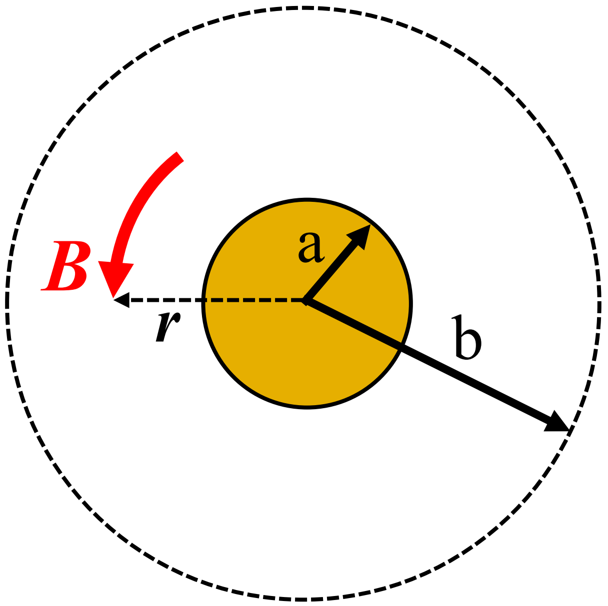
$$\Phi = \oiint \vec{B} \cdot d\vec{s}$$

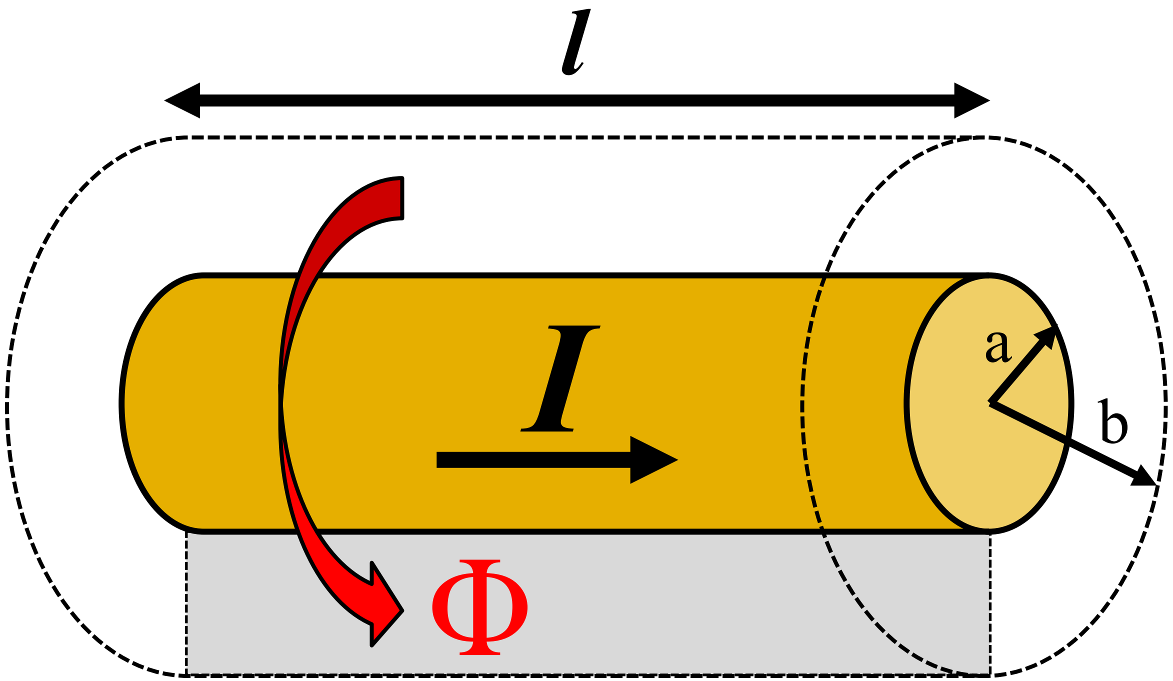
Integrate over surface of interest to obtain total magnetic flux

$$\mathrm{L} = \frac{\Phi}{\mathrm{I}}$$

Divide magnetic flux by current to obtain inductance

Coaxial Cable





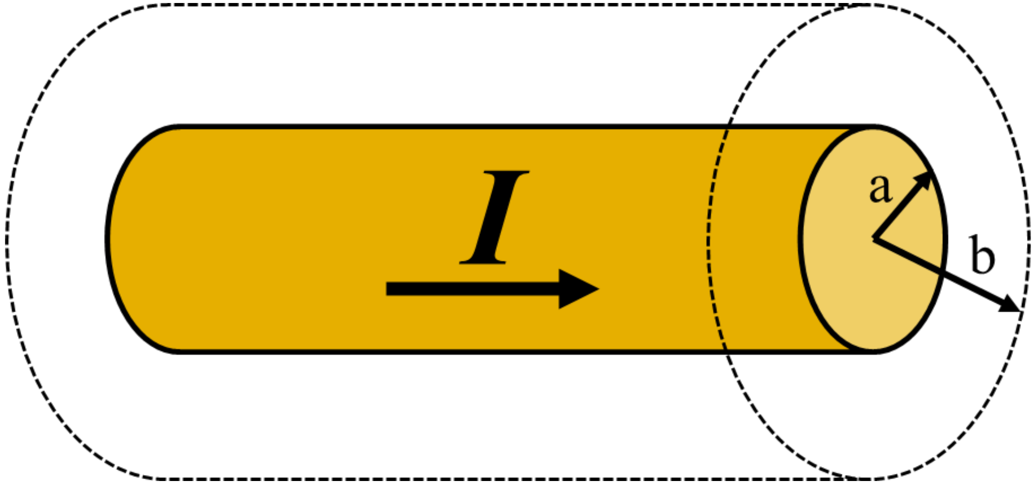
$$B = \frac{\mu I}{2\pi r}$$

$$\Phi = \frac{\mu I l}{2\pi}\int_a^b \frac{dr}{r} \qquad \Phi_l = \frac{\Phi}{l} = \frac{\mu I}{2\pi}\ln\left(\frac{b}{a}\right)$$

$$L_l = \frac{\Phi_l}{I} \qquad \boxed{L_l = \frac{\mu}{2\pi}\ln\left(\frac{b}{a}\right)}$$

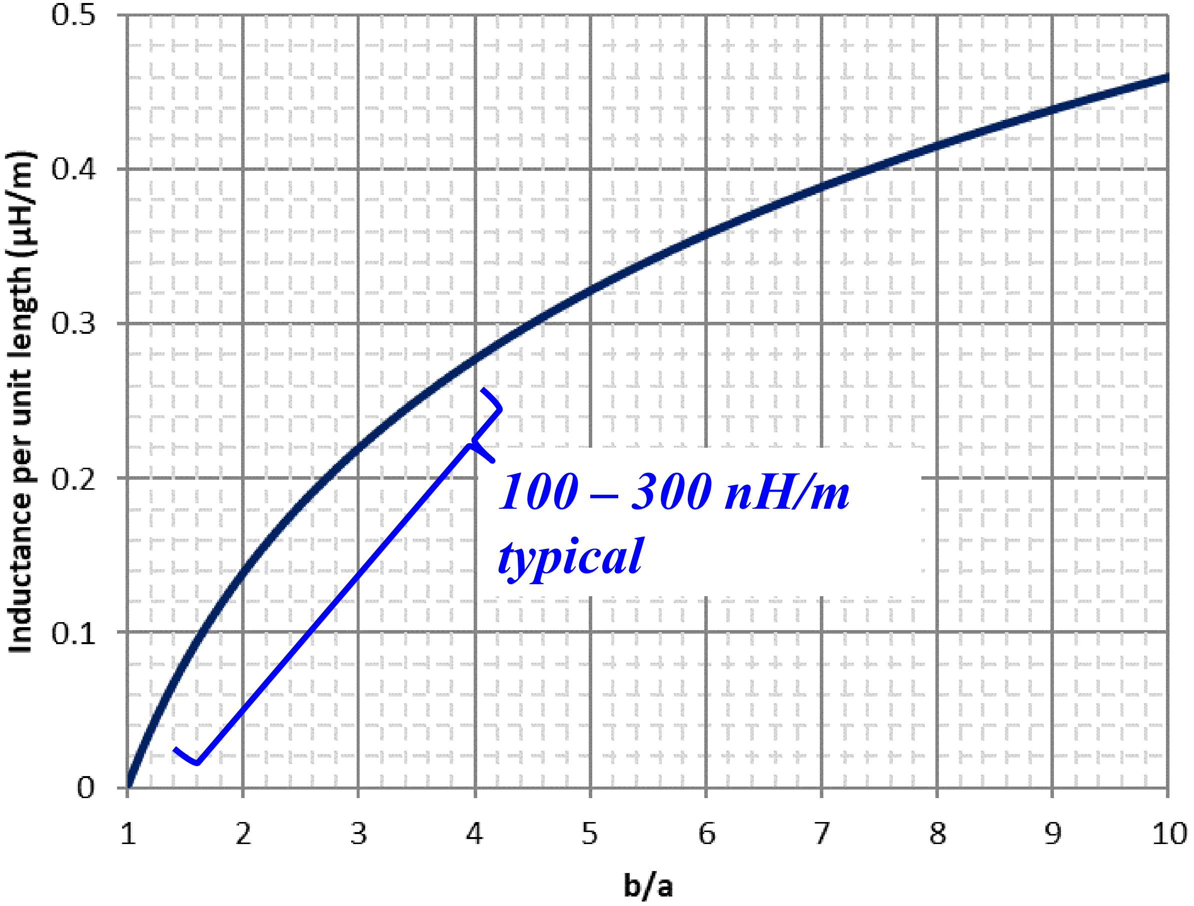
Inductance per unit length for coaxial cable

Coaxial Cable (cont.)

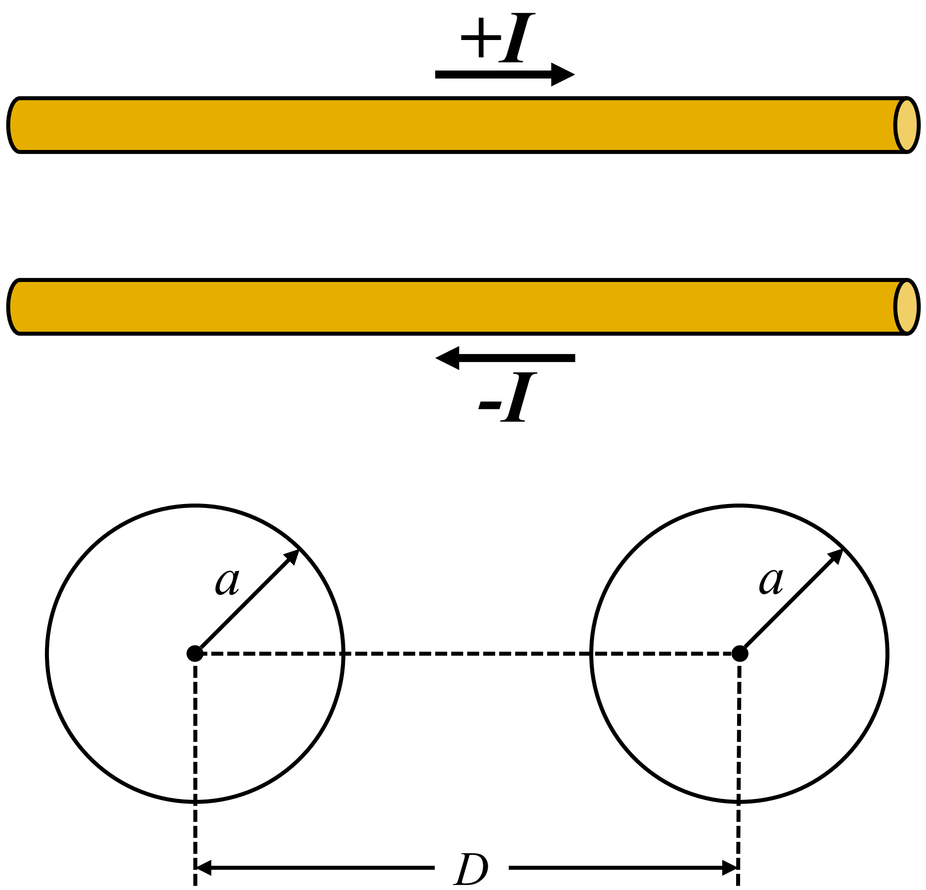


$$L_l = \frac{\mu}{2\pi} \ln\left(\frac{b}{a}\right)$$

$\mu = \mu_r \mu_0 \quad \mu_r = 1$



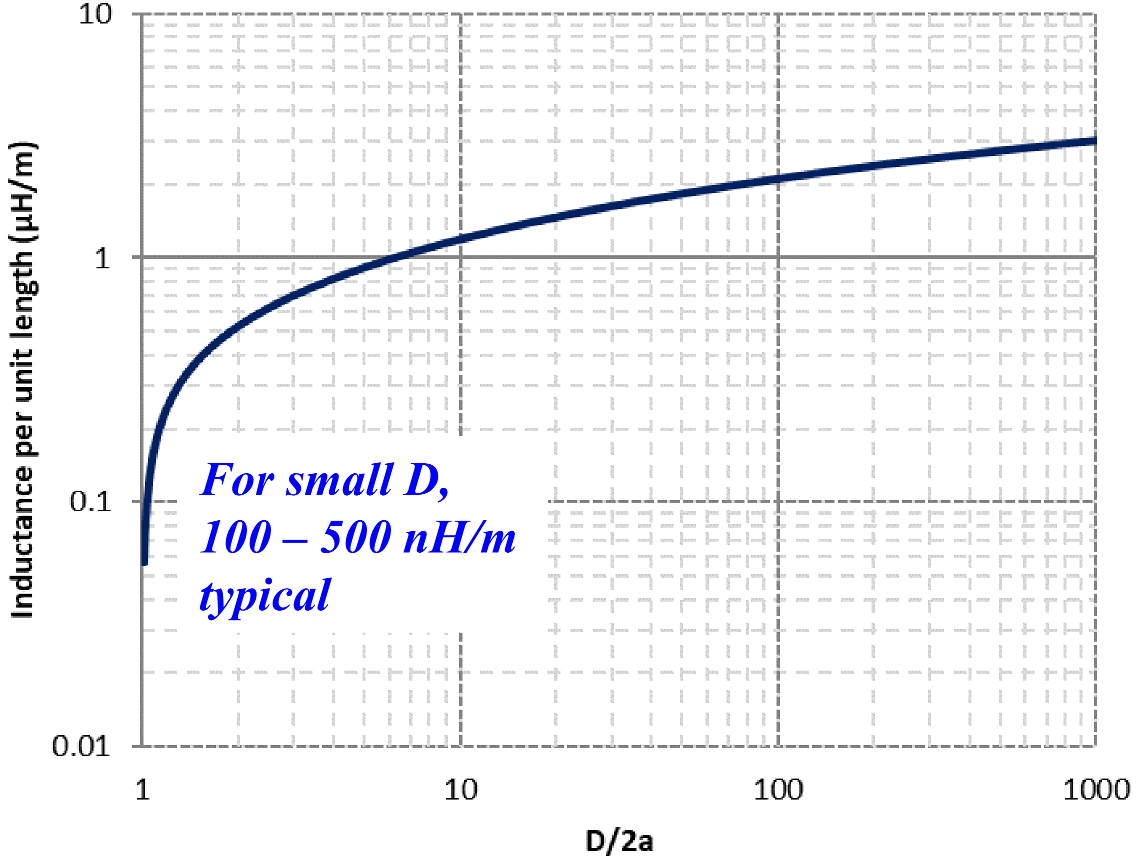
Parallel Wires



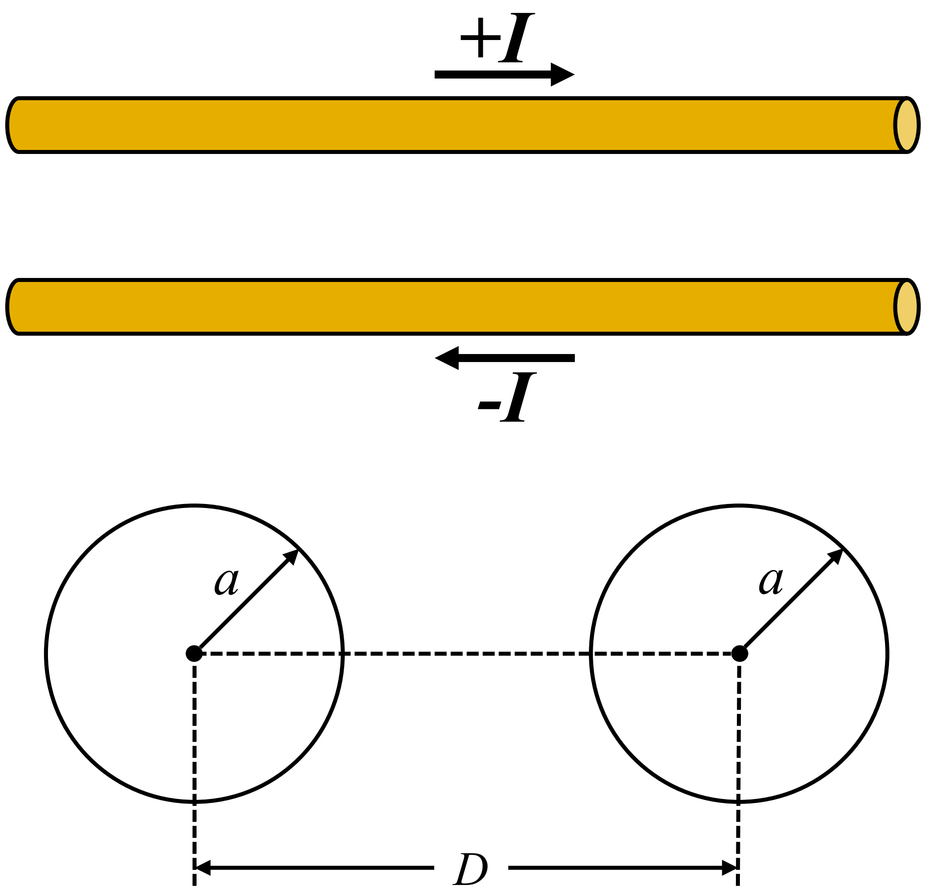
Derived in backup slides

$$L_l = \frac{\mu}{\pi} \cdot \ln\left[\left(\frac{D}{2a}\right) + \sqrt{\left(\frac{D}{2a}\right)^2 - 1}\right]$$

$\mu = \mu_r \mu_0 \quad \mu_r = 1$

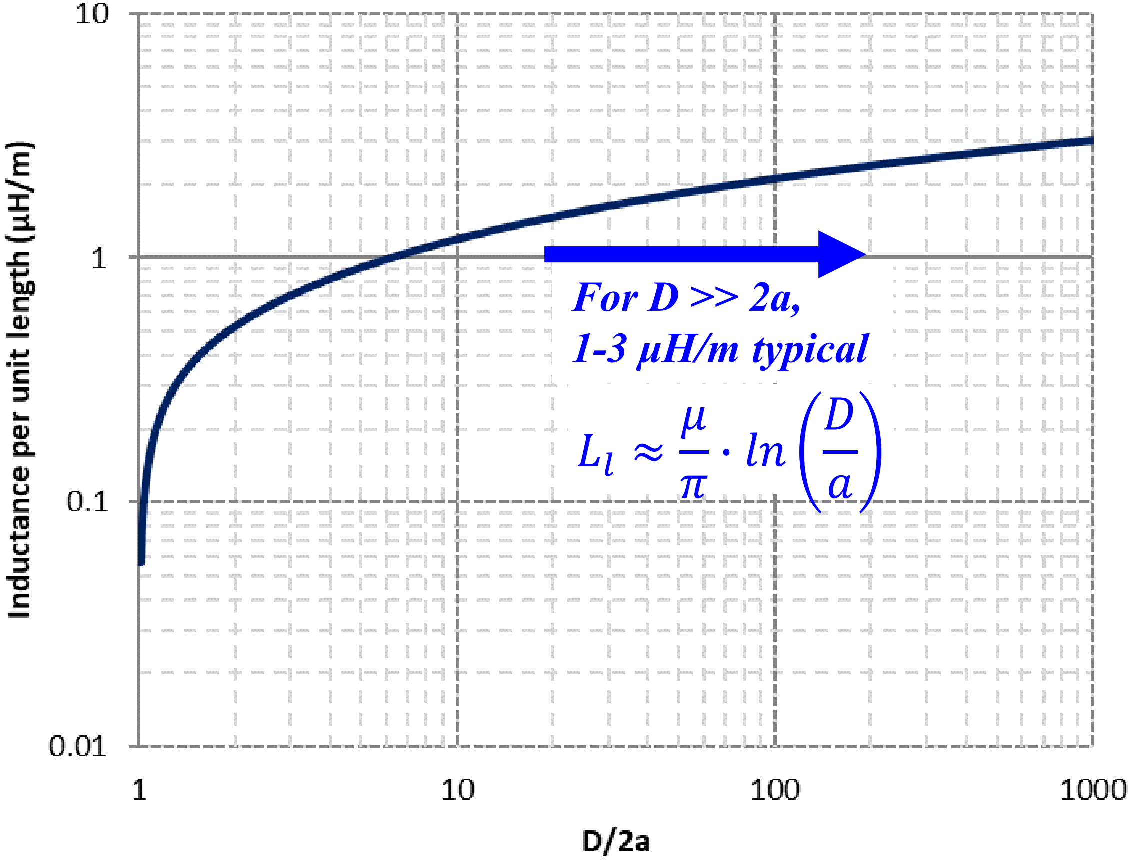


Parallel Wires (cont.)



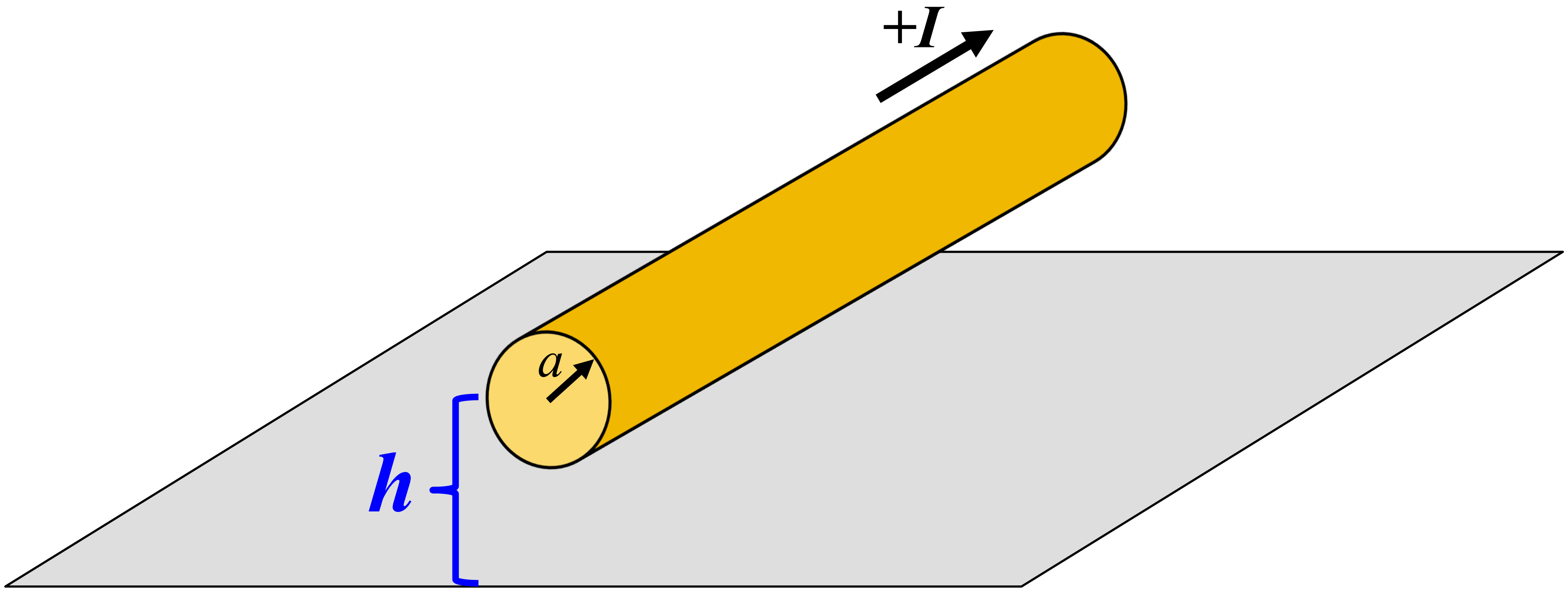
$$L_l = \frac{\mu}{\pi} \cdot \ln\left[\left(\frac{D}{2a}\right) + \sqrt{\left(\frac{D}{2a}\right)^2 - 1}\right]$$

$\mu = \mu_r \mu_0 \quad \mu_r = 1$



$$L_l \approx \frac{\mu}{\pi} \cdot ln\left(\frac{D}{a}\right)$$

Wire Above Ground Plane



Wire Above Ground Plane (cont.)

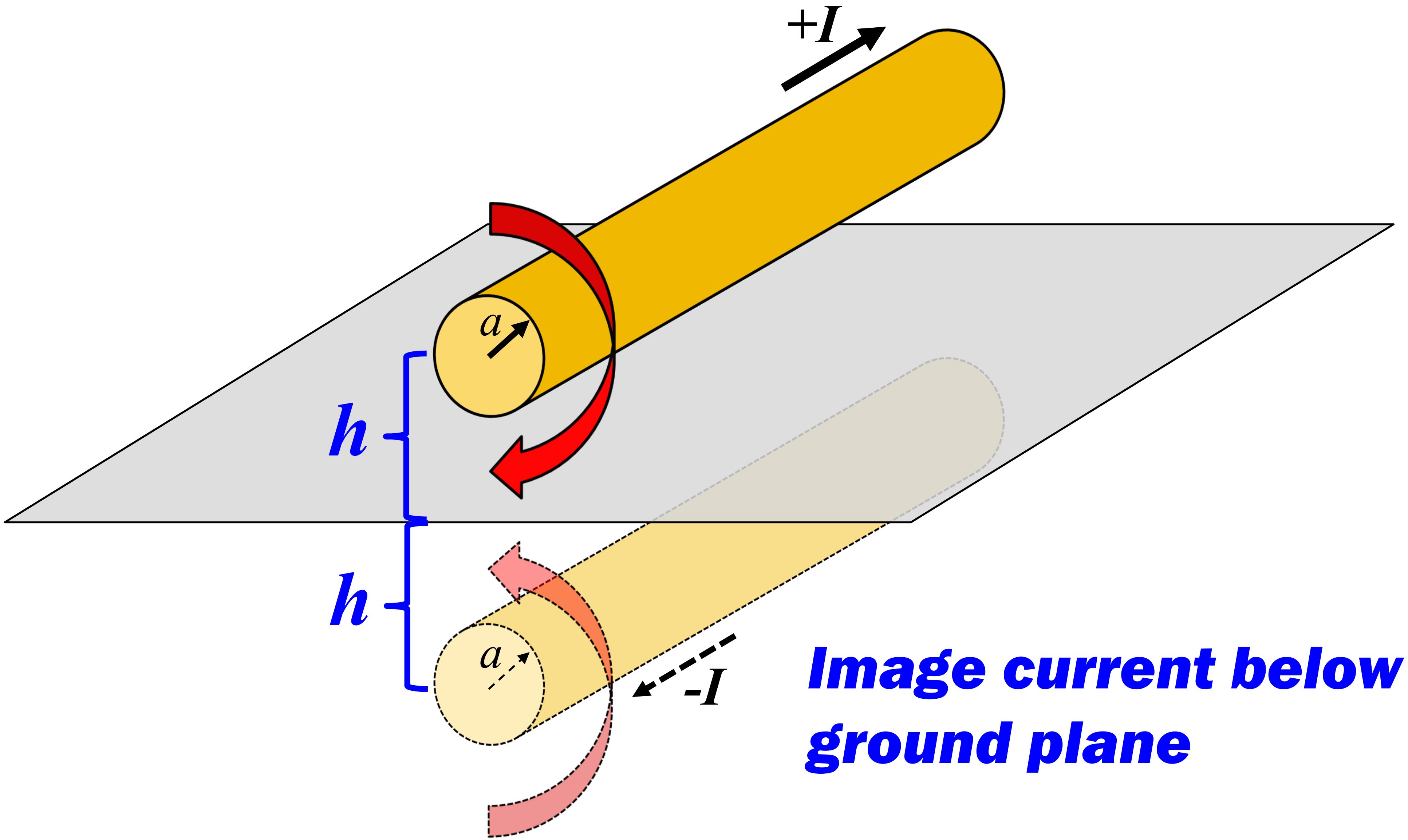
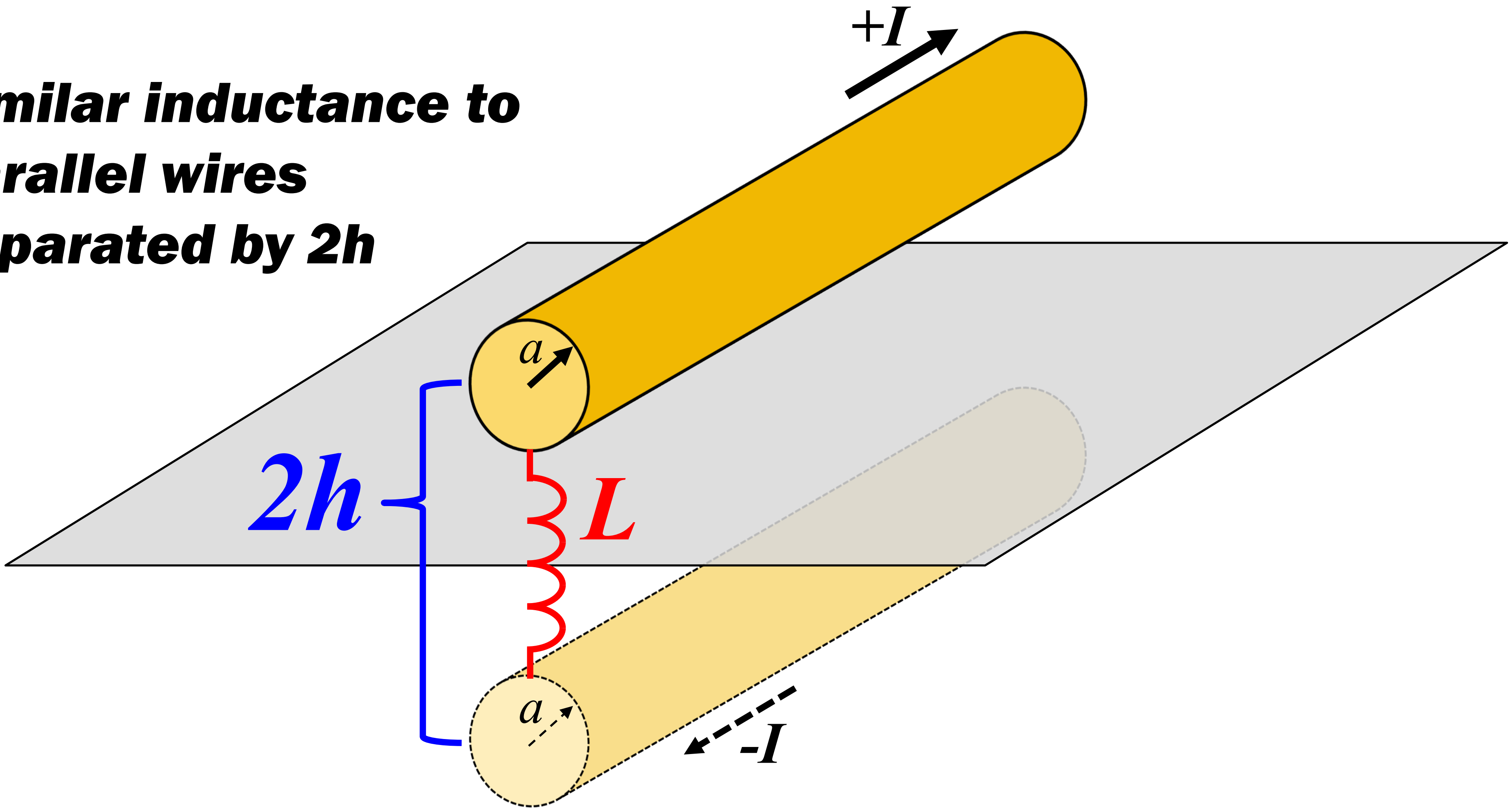


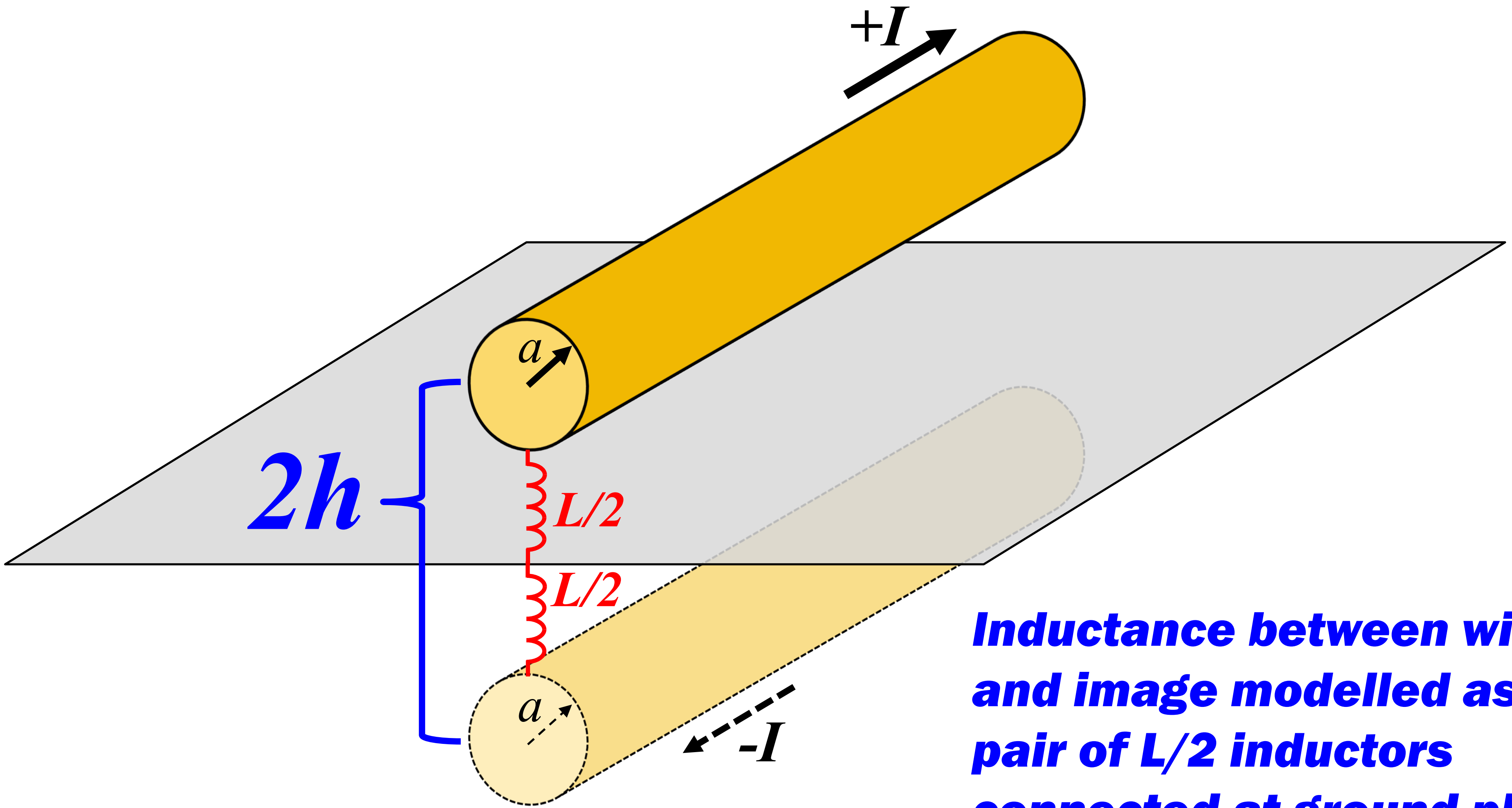
Image current below ground plane

Wire Above Ground Plane (cont.)

Similar inductance to parallel wires separated by 2h

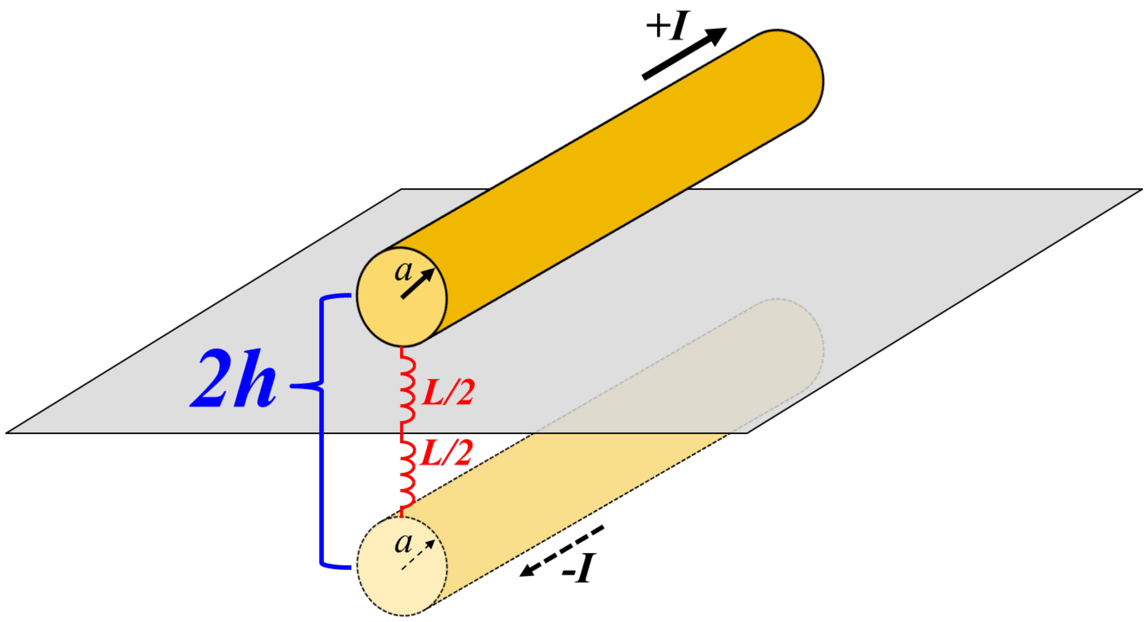


Wire Above Ground Plane (cont.)



Inductance between wire and image modelled as pair of L/2 inductors connected at ground plane

Wire Above Ground Plane (cont.)



Parallel wires:

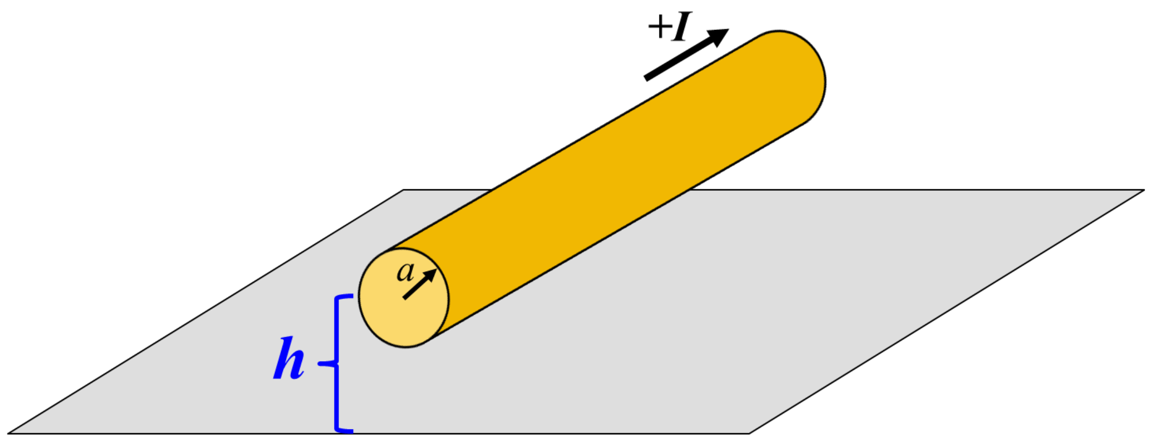
$$L_l = \frac{\mu}{\pi} \cdot \ln\left[\left(\frac{D}{2a}\right) + \sqrt{\left(\frac{D}{2a}\right)^2 - 1}\right]$$

Substitute 2h for D
Divide entire expression by 2
(Half of inductance of equivalent parallel pair)

Wire above ground plane:

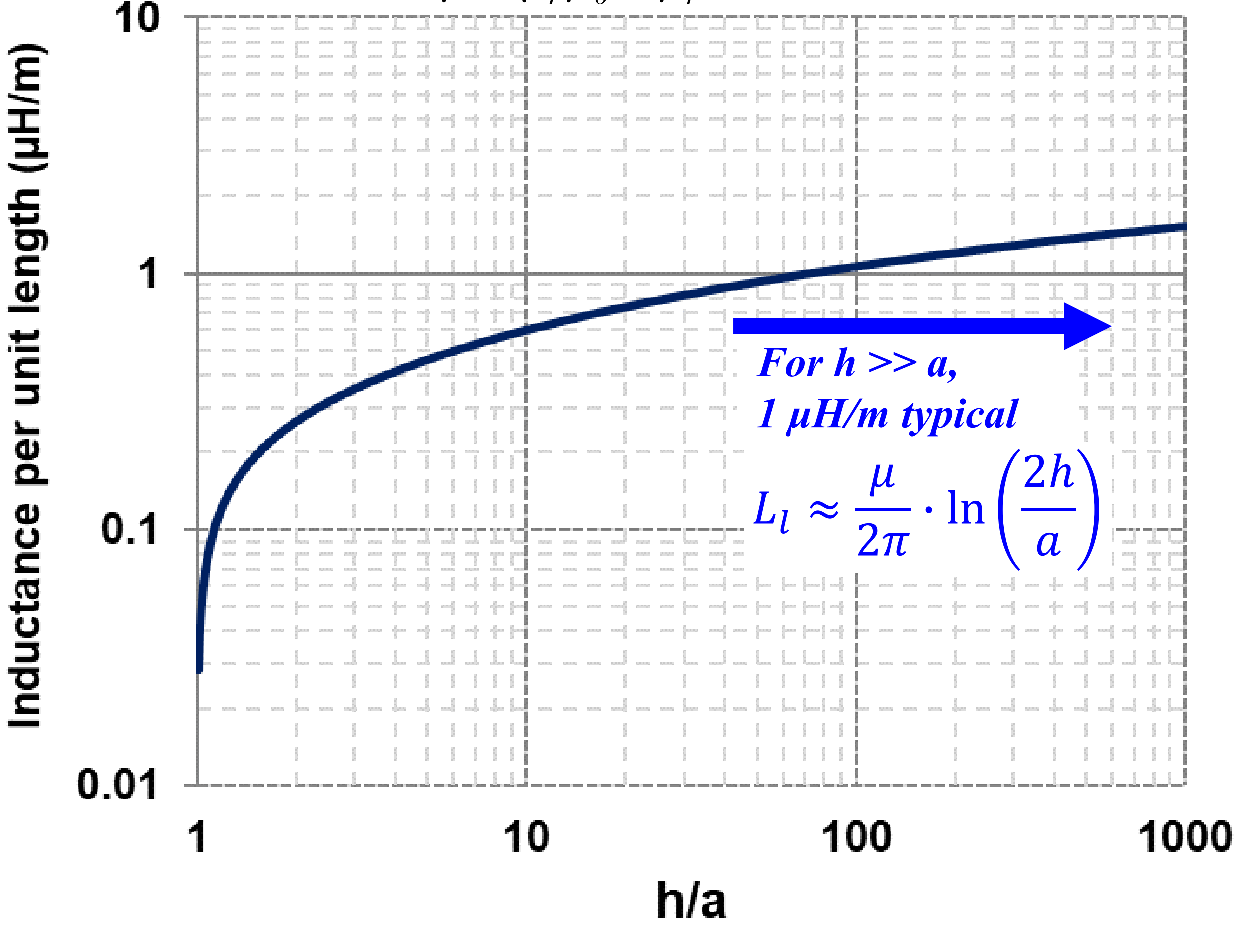
$$L_l = \frac{\mu}{2\pi} \cdot \ln\left[\left(\frac{h}{a}\right) + \sqrt{\left(\frac{h}{a}\right)^2 - 1}\right]$$

Wire Above Ground Plane (cont.)

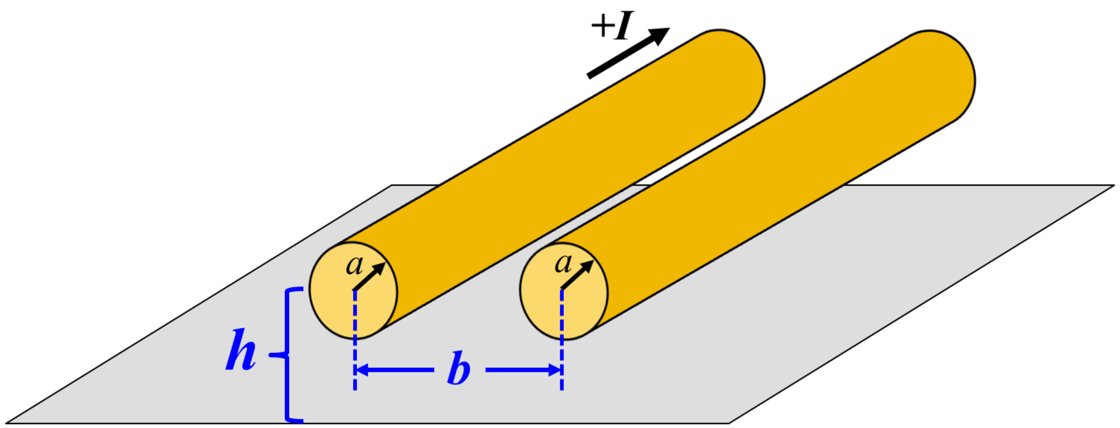


$$L_l = \frac{\mu}{2\pi} \cdot \ln\left[\left(\frac{h}{a}\right) + \sqrt{\left(\frac{h}{a}\right)^2 - 1}\right]$$

$\mu = \mu_r \mu_0 \quad \mu_r = 1$



Parallel Wires Above Ground Plane (cont.)



In coupling demonstration:

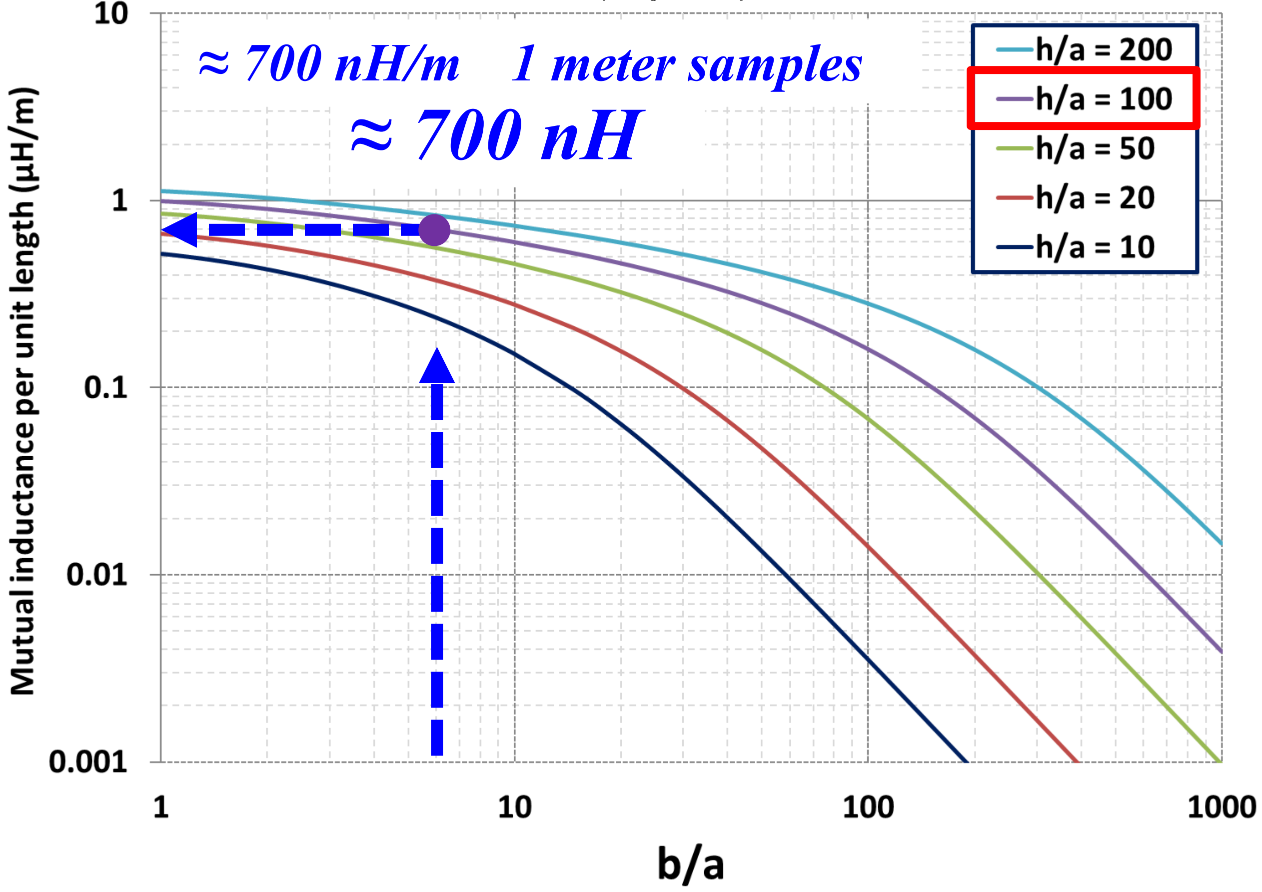
$$\left.\begin{aligned} a &\approx 0.5\ mm \\ b &\approx 3\ mm \end{aligned}\right\} \frac{b}{a} \approx 6$$

$$h \approx 5\ cm \longrightarrow \frac{h}{a} \approx 100$$

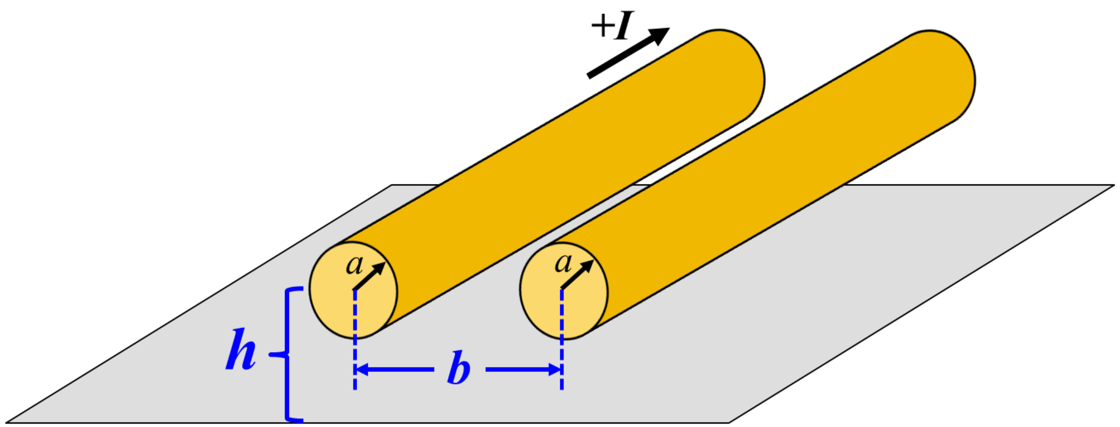
Mutual (coupling) inductance (derived in backup)

$$L_M = \frac{\mu}{4\pi} \cdot \ln\left[\frac{\left(\frac{b}{a}\right)^2 + \left(\frac{2h}{a} - 1\right)^2}{\left(\frac{b}{a}\right)^2 + 1}\right]$$

$$\mu = \mu_r \mu_0 \quad \mu_r = 1$$



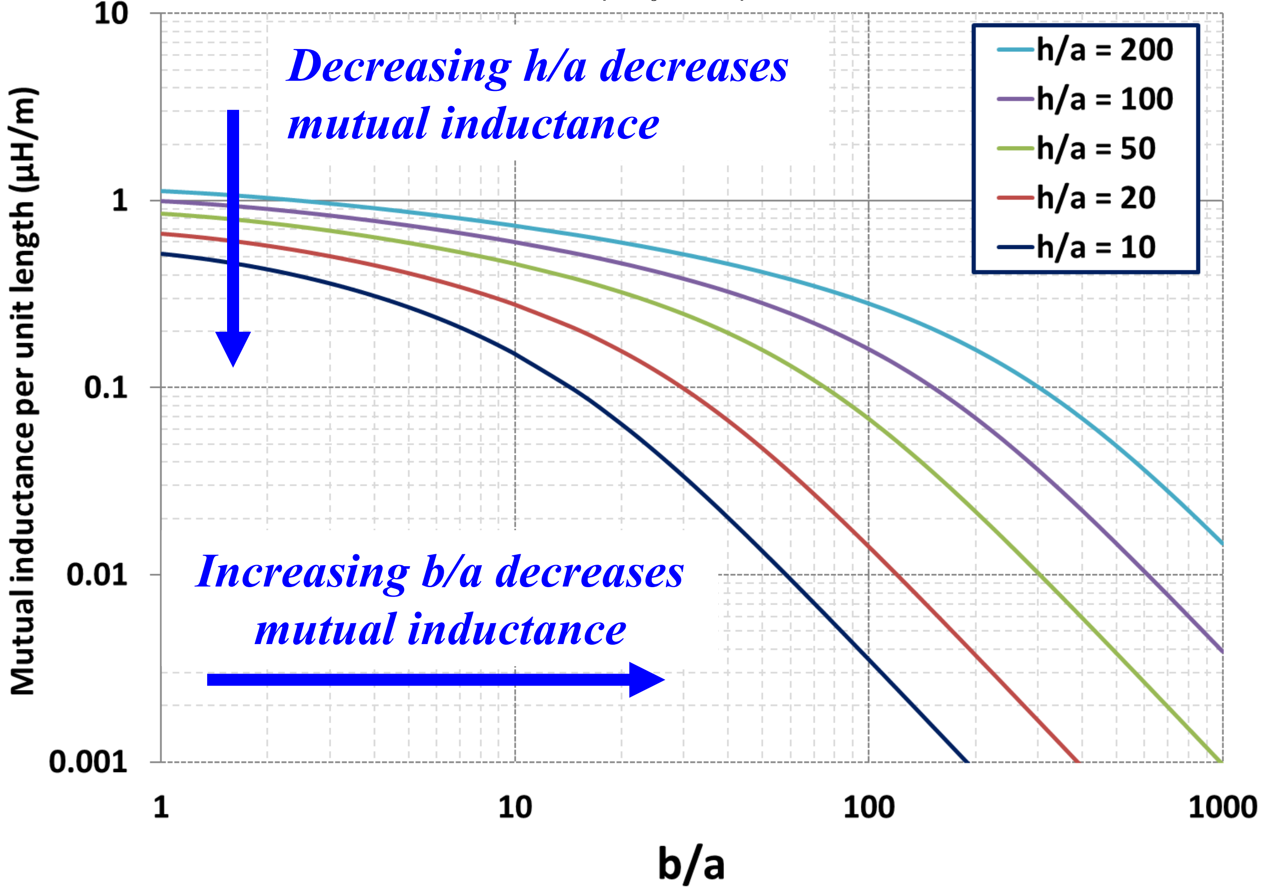
Parallel Wires Above Ground Plane (cont.)



Mutual (coupling) inductance (derived in backup)

$$L_M = \frac{\mu}{4\pi} \cdot \ln\left[\frac{\left(\frac{b}{a}\right)^2 + \left(\frac{2h}{a} - 1\right)^2}{\left(\frac{b}{a}\right)^2 + 1}\right]$$

$\mu = \mu_r \mu_0 \quad \mu_r = 1$

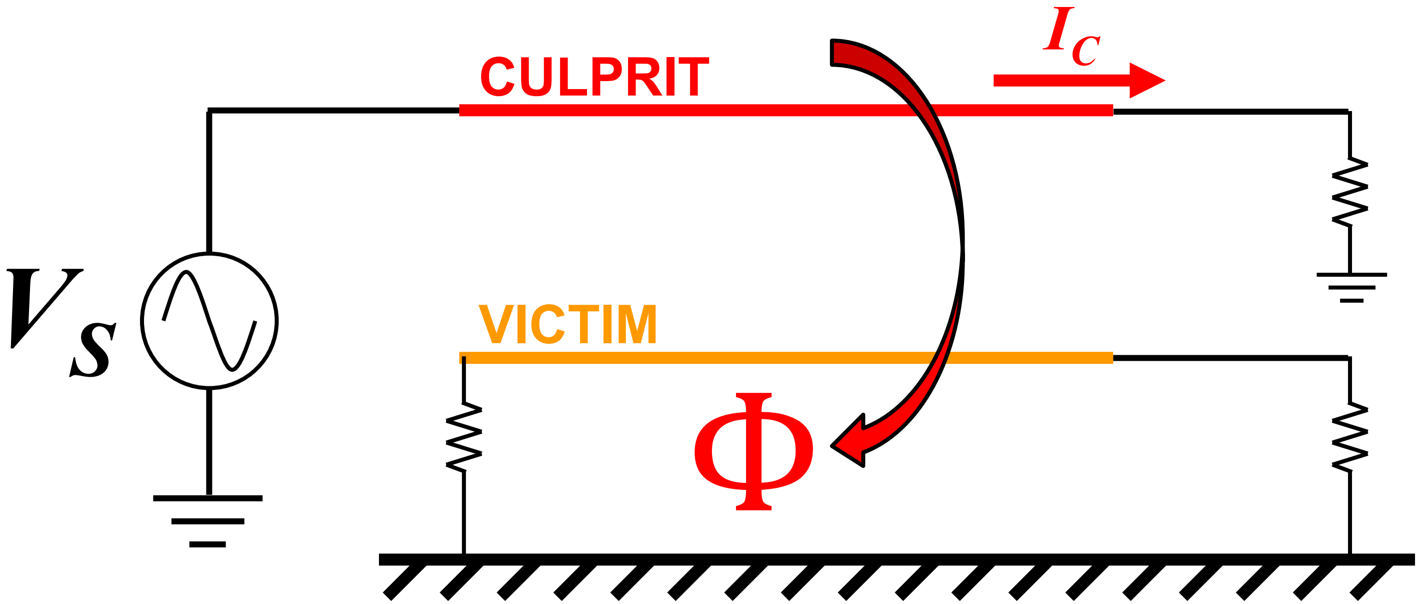


Also worth noting:

Increasing wire separation (b/a) decreases mutual inductance and coupling

Reducing height above ground plane (h/a) decreases mutual inductance and coupling

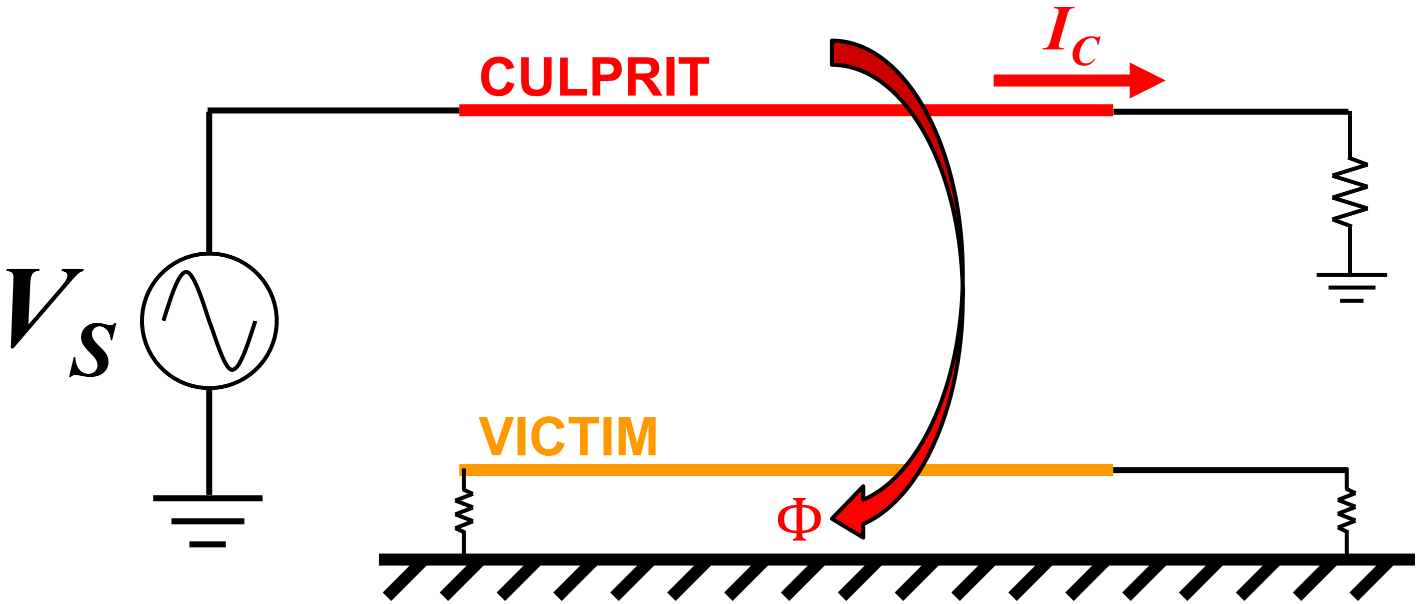
Coupling and Loop Area



$$\Phi = L_M I_C = \oiint \vec{B} \cdot d\vec{s}$$

Magnetic coupling occurs through loop
NOT directly to conductors per se
Depends on loop area

Coupling and Loop Area (cont.)



$$\Phi = L_M I_C = \oiint \vec{B} \cdot d\vec{s}$$

Reduce loop area
(e.g. place cables near ground plane)
→ reduce coupling

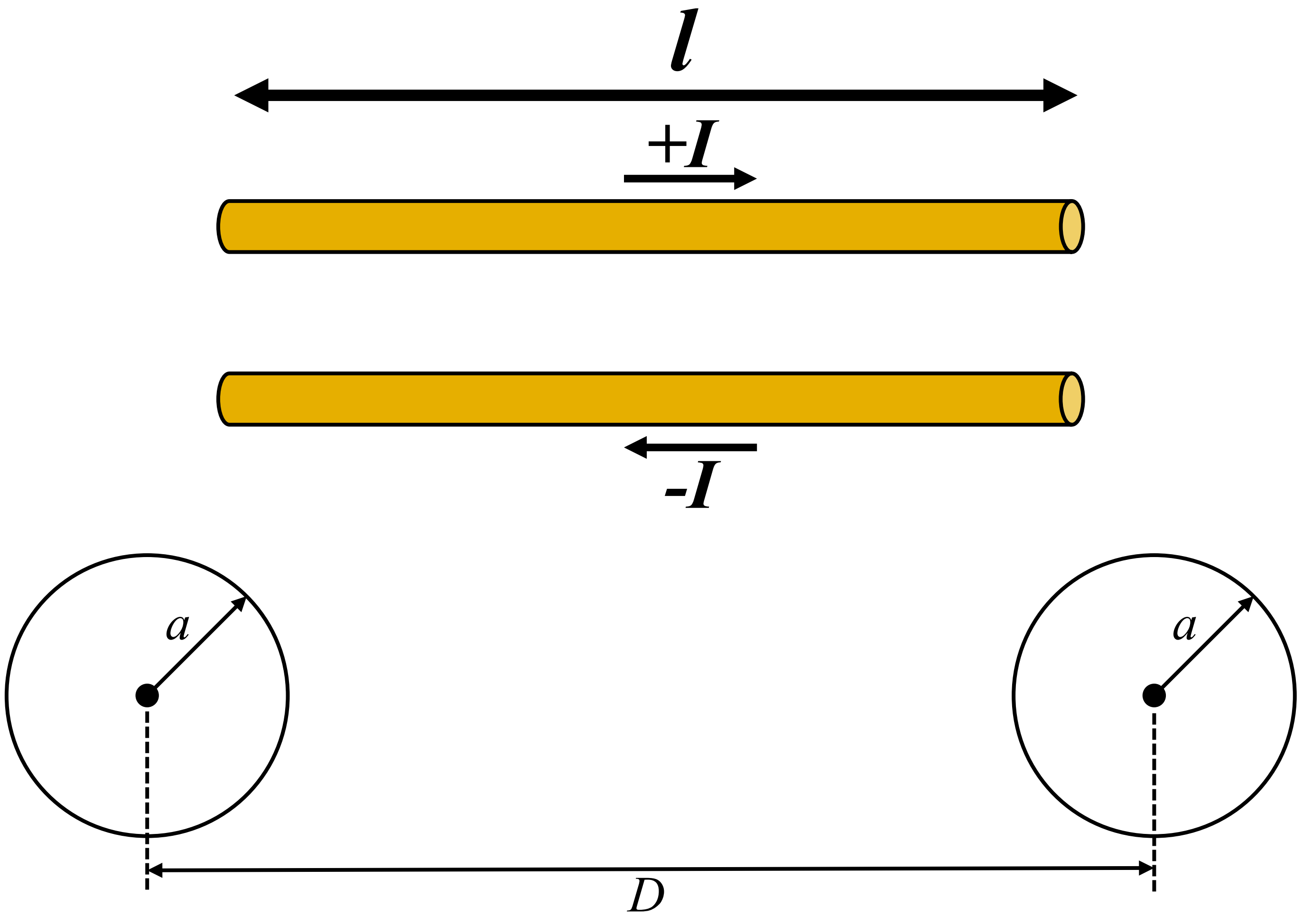
Magnetic Field, Current, and Inductance Summary

- Magnetic field is directly related to familiar concepts of current and inductance
- Current produces magnetic field and magnetic flux
- Magnetic flux coupled into loop induces potential (electromotive force, a.k.a. “emf”)
- Inductance is defined as total flux divided by current causing it and is present whether or not it appears on your schematic
- Coupling depends directly on loop area
- Reduce loop area → reduce coupling

Magnetic Field, Current, and Inductance

BACKUP

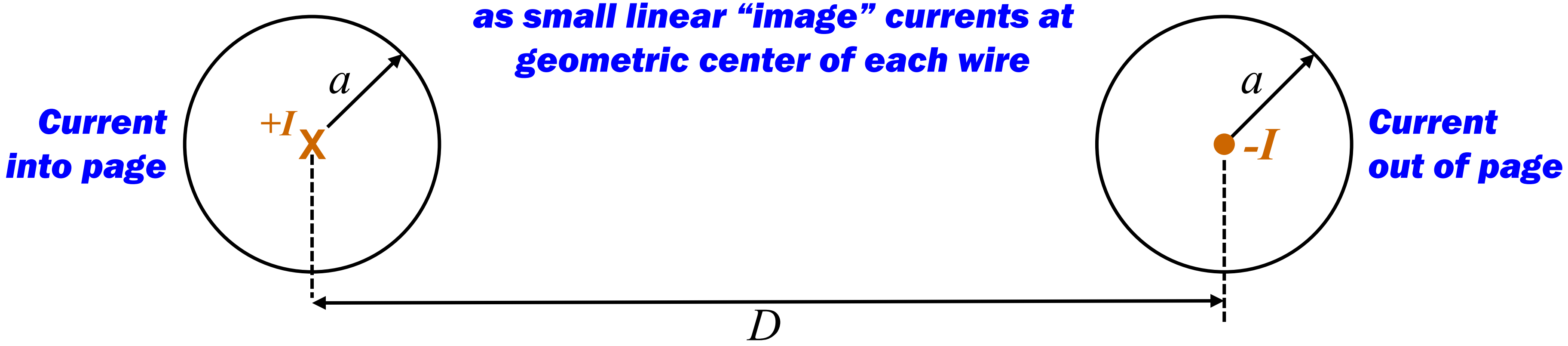
Parallel Wires



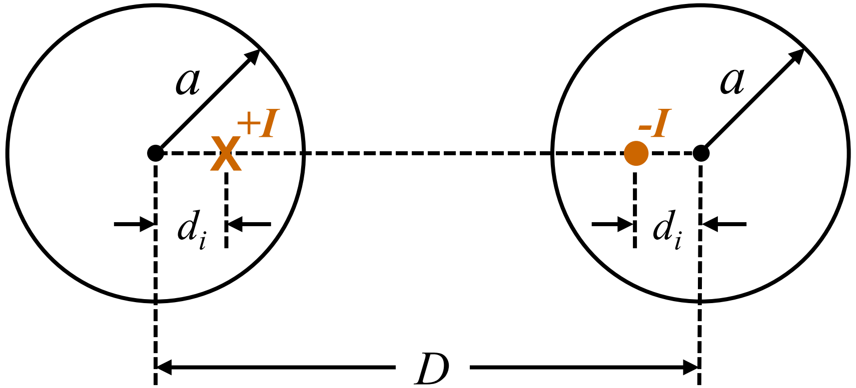
Details in "Fundamentals of Electromagnetics" video:
Magnetic Field, Current, and Inductance – Part 2

Parallel Wires (cont.)

For D >> a, currents can be modelled as small linear "image" currents at geometric center of each wire

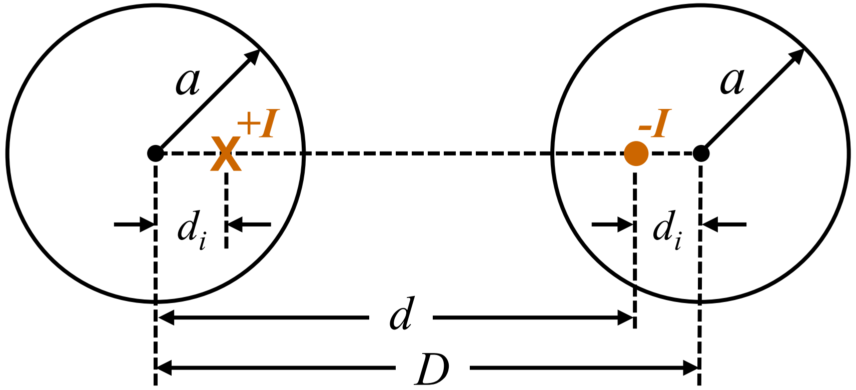


Parallel Wires (cont.)



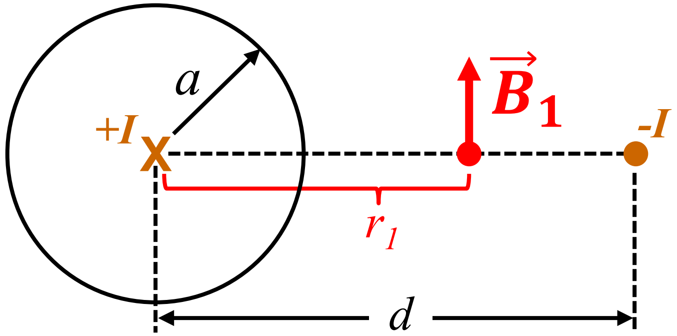
For small D, “image” currents are offset from geometric centers

Parallel Wires (cont.)



$$d = D - d_i$$

Parallel Wires (cont.)

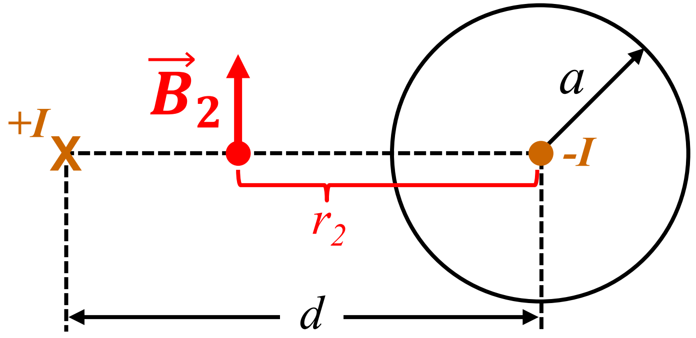


$$B_1 = \frac{\mu I}{2\pi r_1}$$

$$\Phi_1 = \frac{\mu I l}{2\pi}\int_a^d \frac{dr_1}{r_1}$$

$$\Phi_1 = \frac{\mu I l}{2\pi}\ln\left(\frac{d}{a}\right)$$

Parallel Wires (cont.)

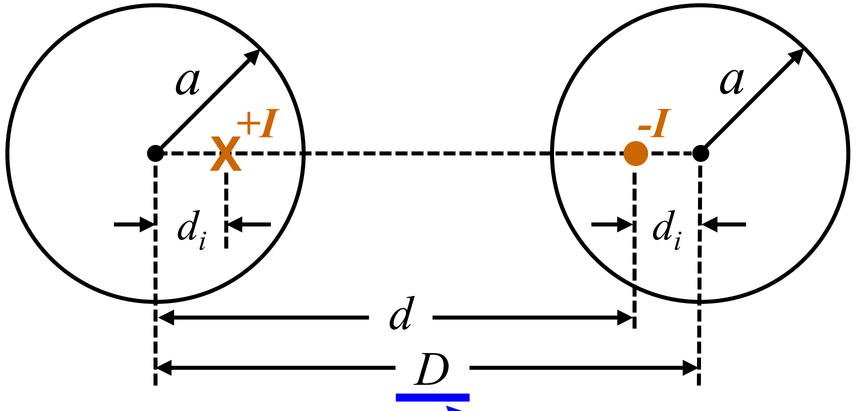


$$B_2 = \frac{\mu I}{2\pi r_2}$$

$$\Phi_2 = \frac{\mu I l}{2\pi}\int_a^d \frac{dr_2}{r_2}$$

$$\Phi_2 = \frac{\mu I l}{2\pi}\ln\left(\frac{d}{a}\right)$$

Parallel Wires (cont.)



$$\Phi_{TOT} = \Phi_1 + \Phi_2$$

$$\Phi_{TOT} = \frac{\mu I l}{2\pi} \ln\left(\frac{d}{a}\right) + \frac{\mu I l}{2\pi} \ln\left(\frac{d}{a}\right)$$

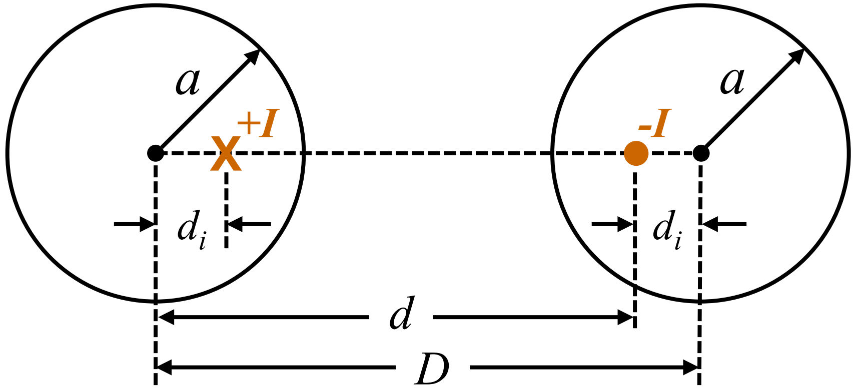
$$\Phi_{TOT} = \frac{\mu I l}{\pi} \ln\left(\frac{d}{a}\right)$$

$$L = \frac{\Phi_{TOT}}{I} = \frac{\mu l}{\pi} \ln\left(\frac{d}{a}\right)$$

$$L_l = \frac{\mu}{\pi} \ln\left(\frac{d}{a}\right)$$

Need this in terms of large D

Parallel Wires (cont.)



$$d = D - d_i$$

$$L_l = \frac{\mu}{\pi}\ln\left(\frac{d}{a}\right)$$

Derived in "Electric Fields" session:

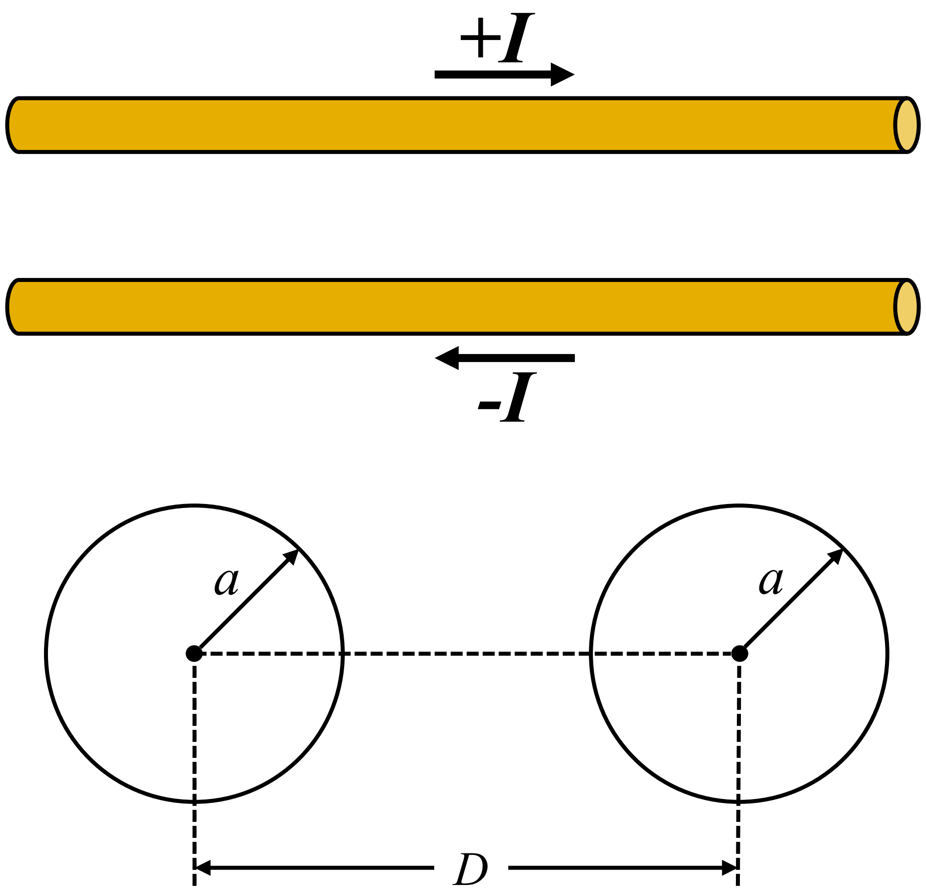
$$d = \frac{D + \sqrt{D^2 - 4a^2}}{2}$$

$$\frac{d}{a} = \left(\frac{D}{2a}\right) + \sqrt{\left(\frac{D}{2a}\right)^2 - 1}$$

$$\boxed{L_l = \frac{\mu}{\pi}\cdot\ln\left[\left(\frac{D}{2a}\right) + \sqrt{\left(\frac{D}{2a}\right)^2 - 1}\right]}$$

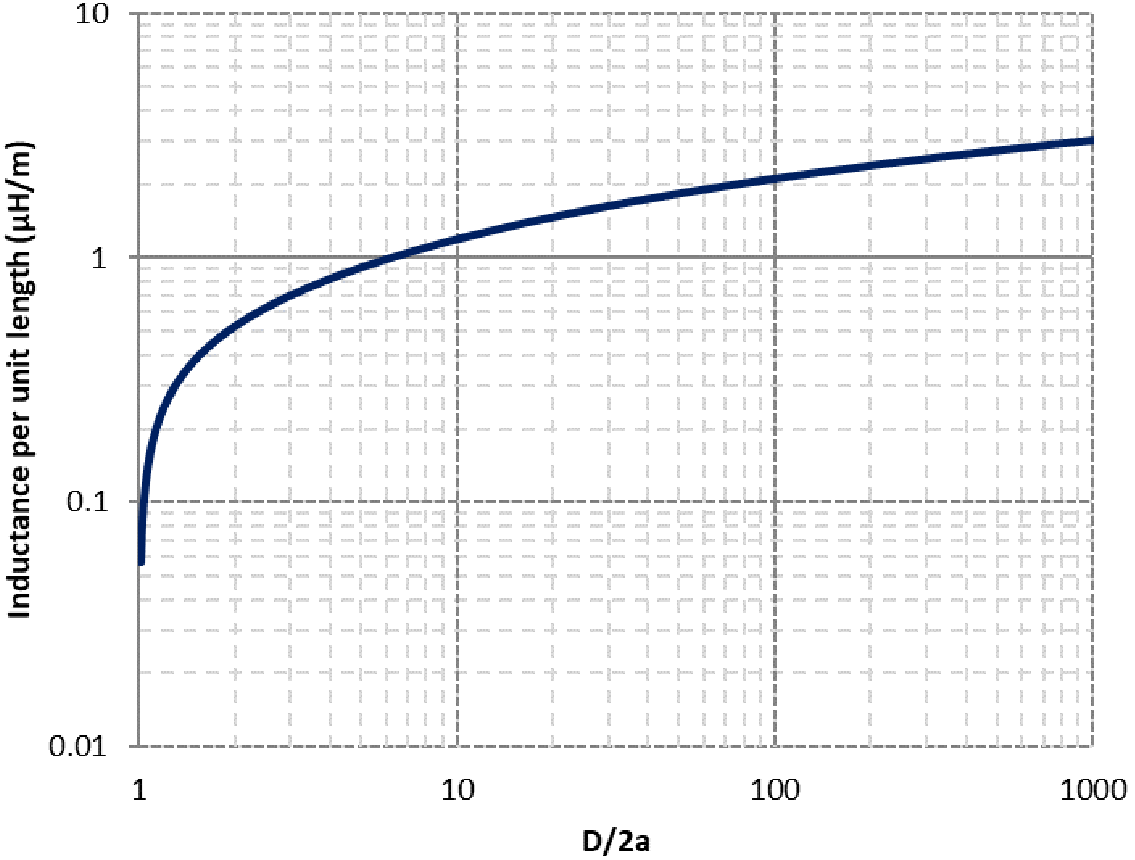
Inductance per unit length
for parallel wires

Parallel Wires (cont.)

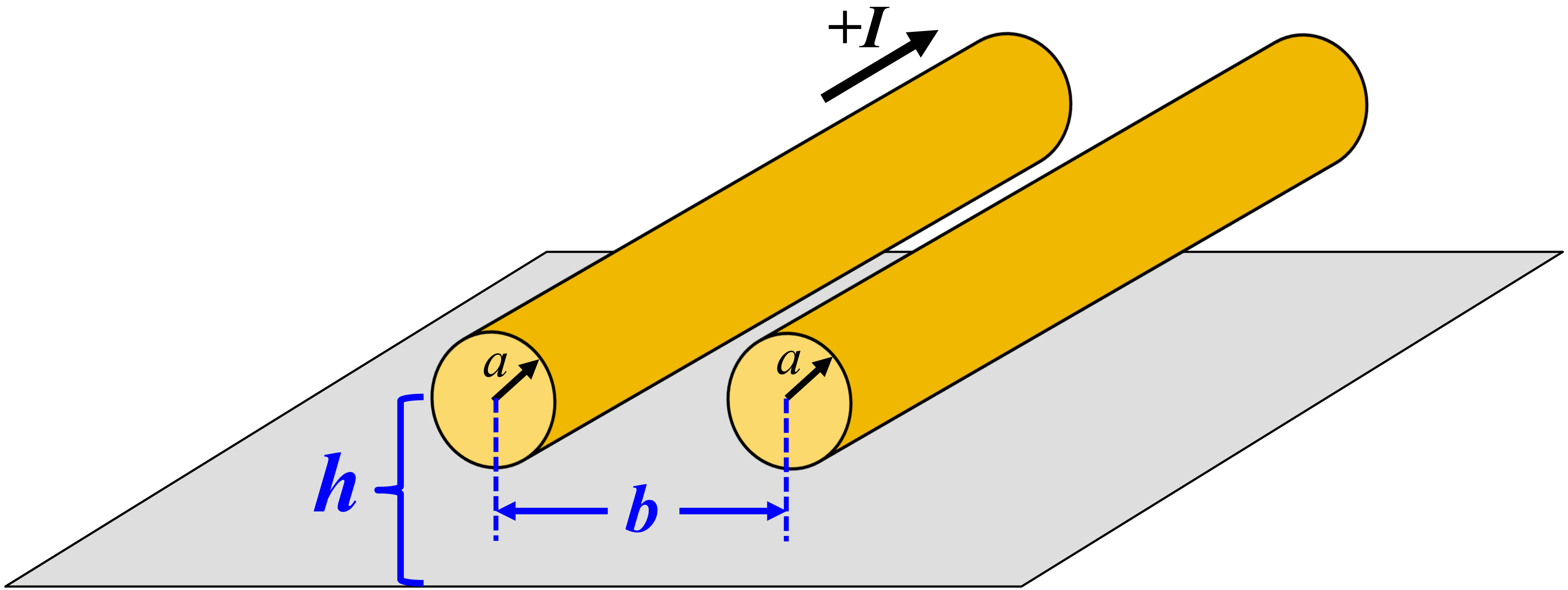


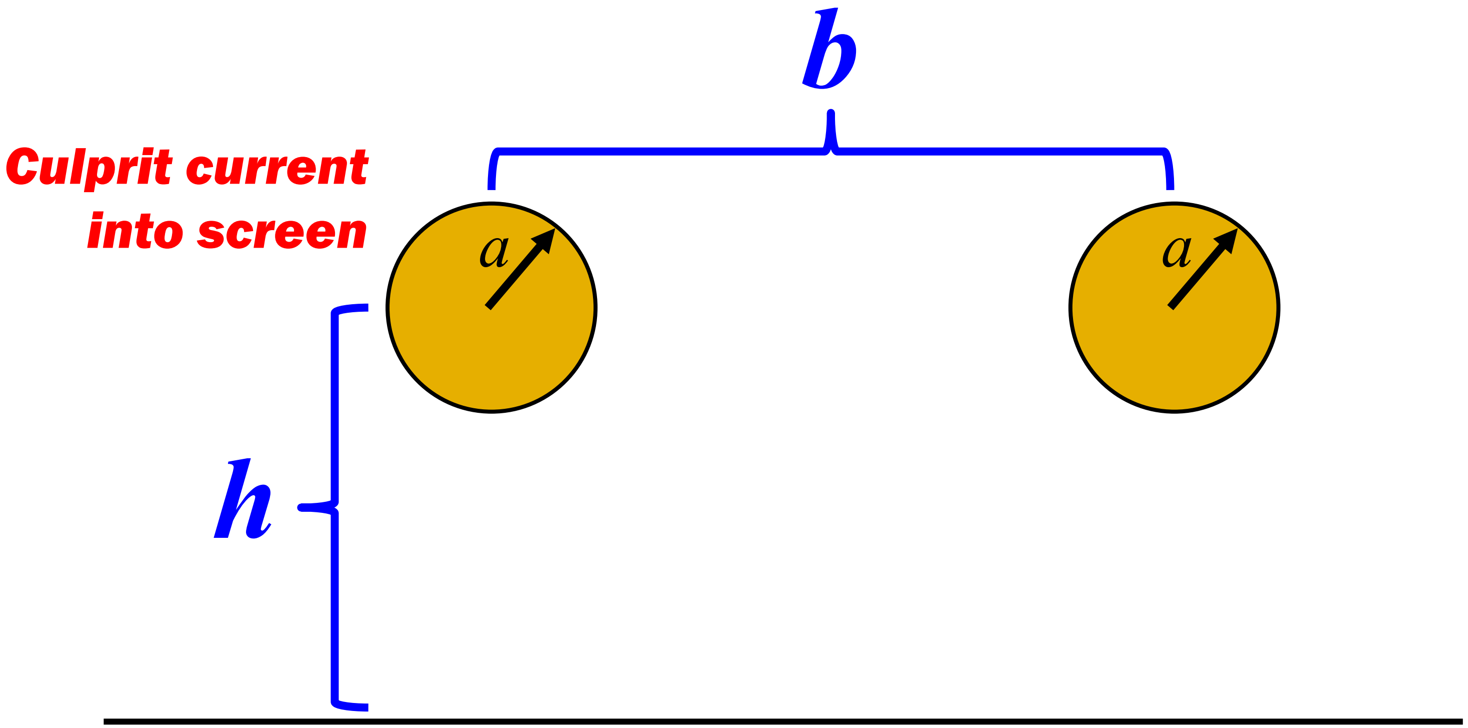
$$L_l = \frac{\mu}{\pi} \cdot \ln\left[\left(\frac{D}{2a}\right) + \sqrt{\left(\frac{D}{2a}\right)^2 - 1}\right]$$

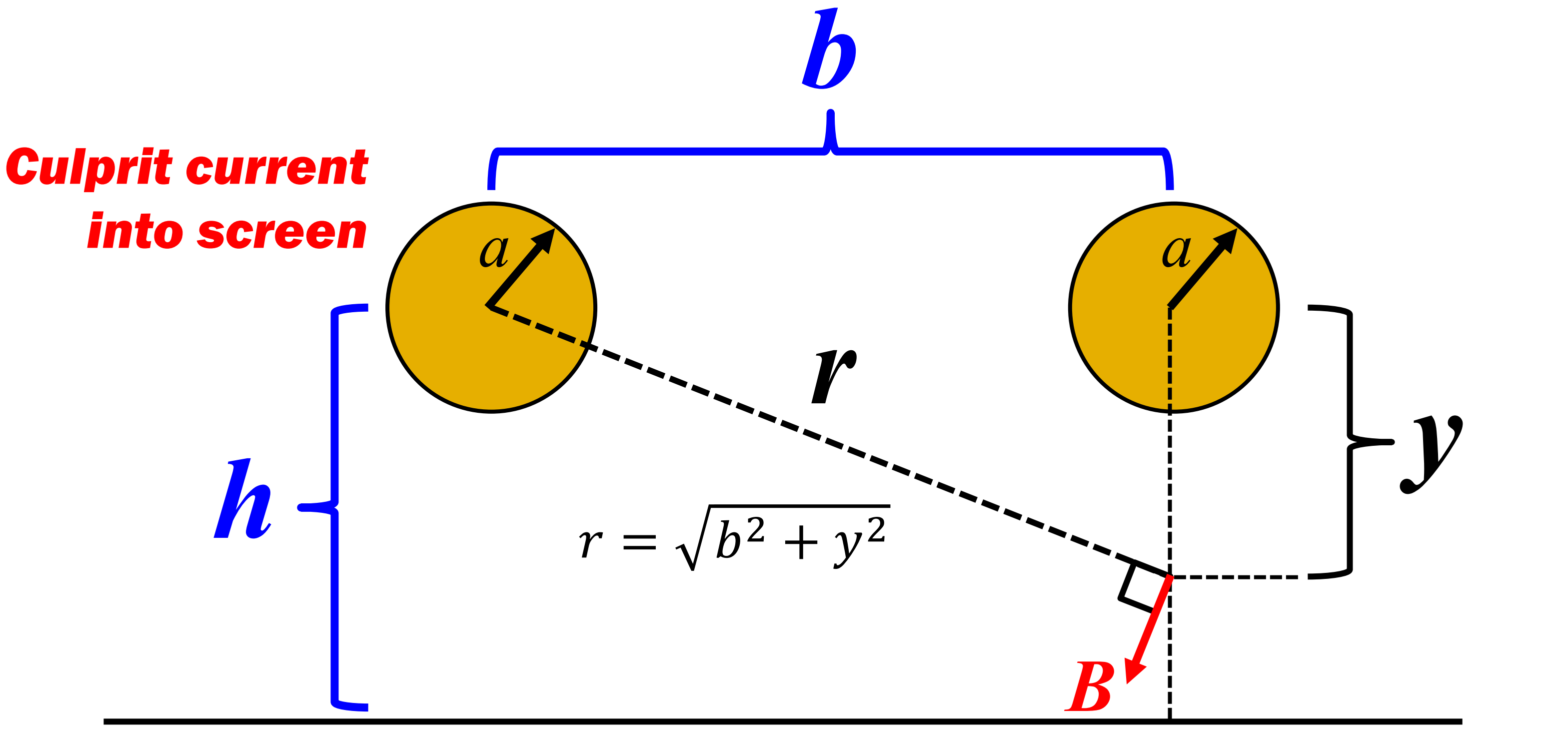
$\mu = \mu_r \mu_0 \quad \mu_r = 1$



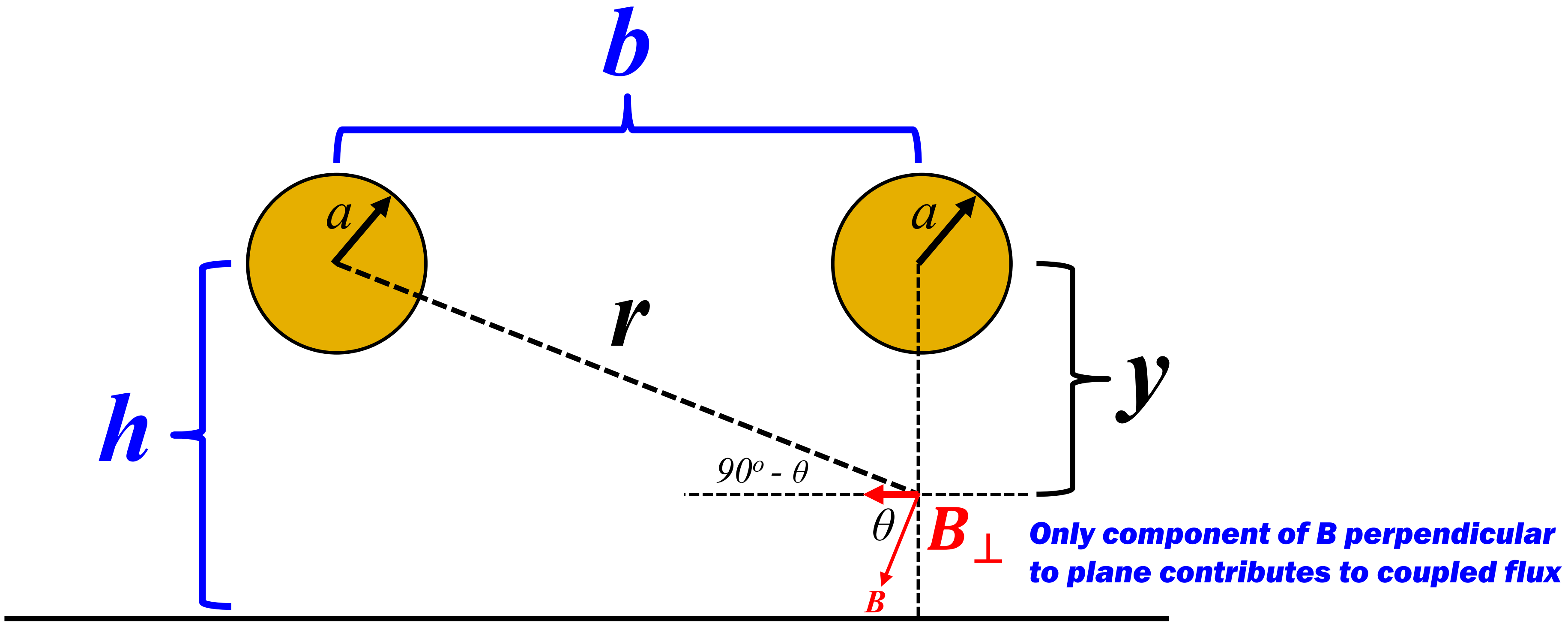
Parallel Wires Above Ground Plane



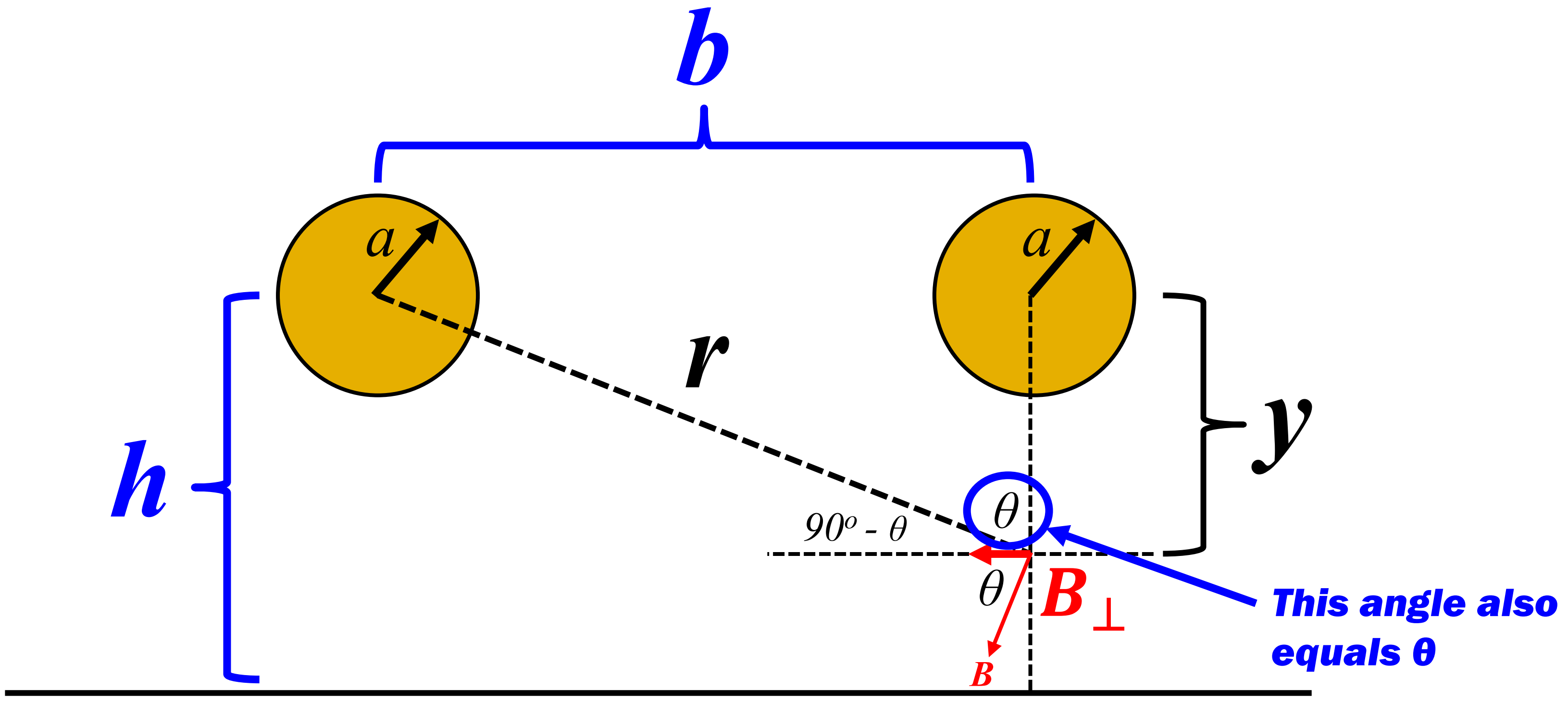

b
Culprit current
into screen
a
a
h



$$B = \frac{\mu I}{2\pi r} = \frac{\mu I}{2\pi\sqrt{b^2 + y^2}}$$



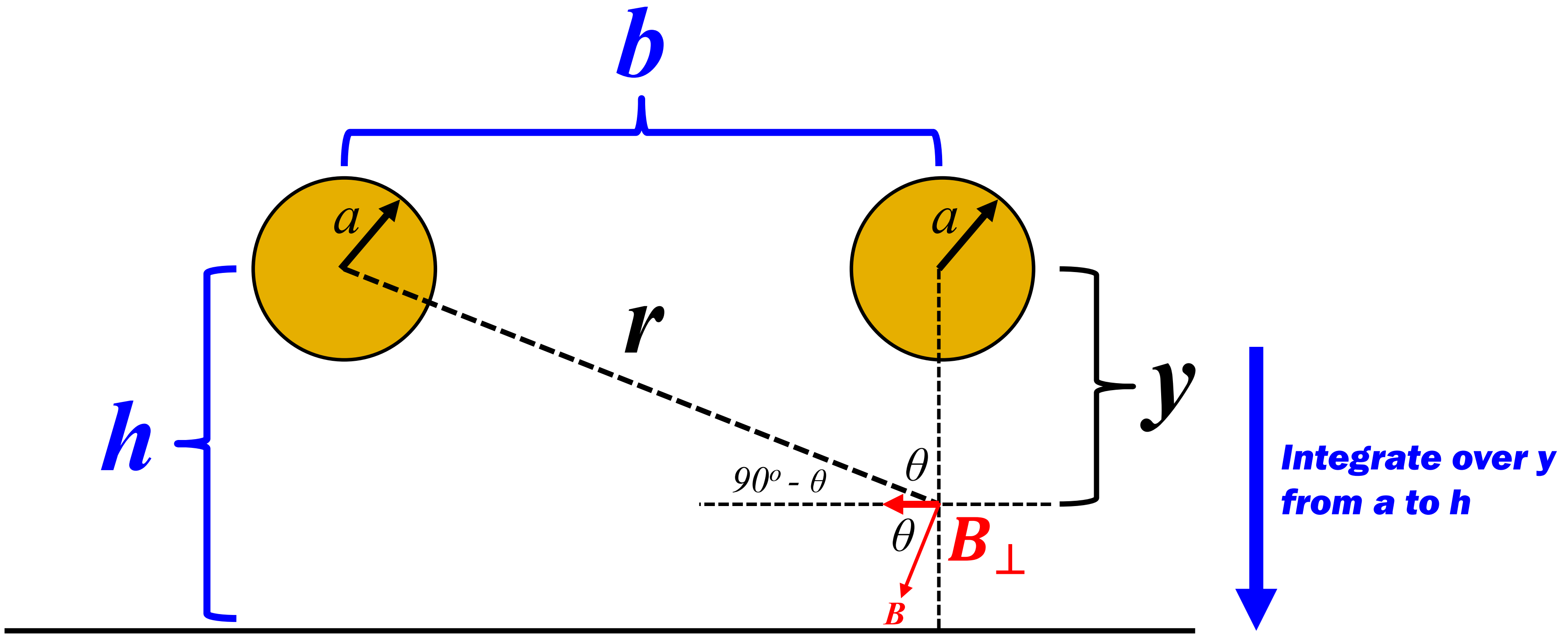
$$B = \frac{\mu I}{2\pi\sqrt{b^2 + y^2}} \qquad B_{\perp} = B\cos\theta$$



$$B = \frac{\mu I}{2\pi\sqrt{b^2 + y^2}}$$

$$B_\perp = B\cos\theta \qquad \cos\theta = \frac{y}{\sqrt{b^2 + y^2}}$$

$$B_\perp = \frac{\mu I}{2\pi} \cdot \frac{y}{(b^2 + y^2)}$$



$$B = \frac{\mu I}{2\pi\sqrt{b^2 + y^2}}$$

$$B_\perp = B\cos\theta \qquad \cos\theta = \frac{y}{\sqrt{b^2 + y^2}}$$

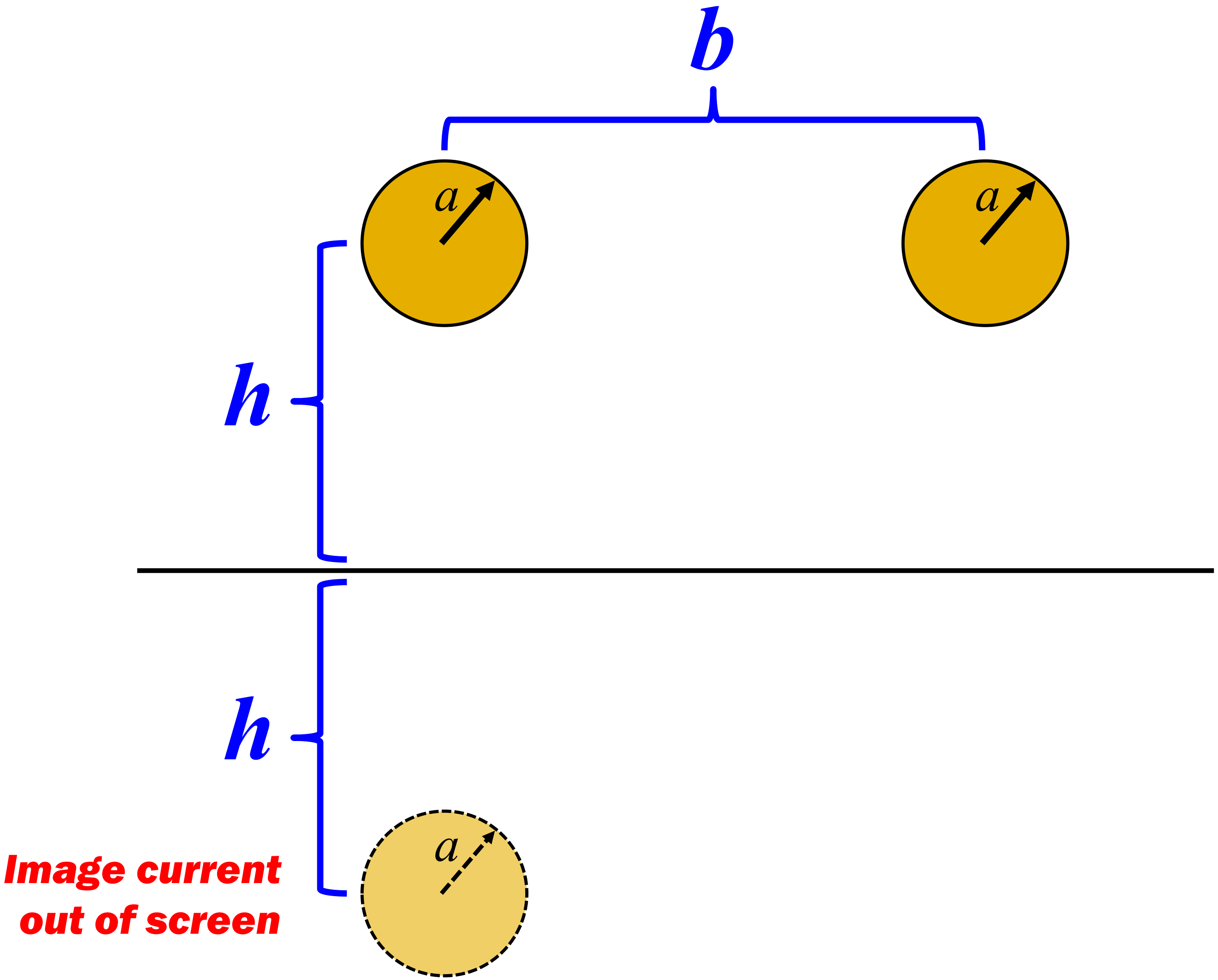
$$B_\perp = \frac{\mu I}{2\pi} \cdot \frac{y}{(b^2 + y^2)}$$

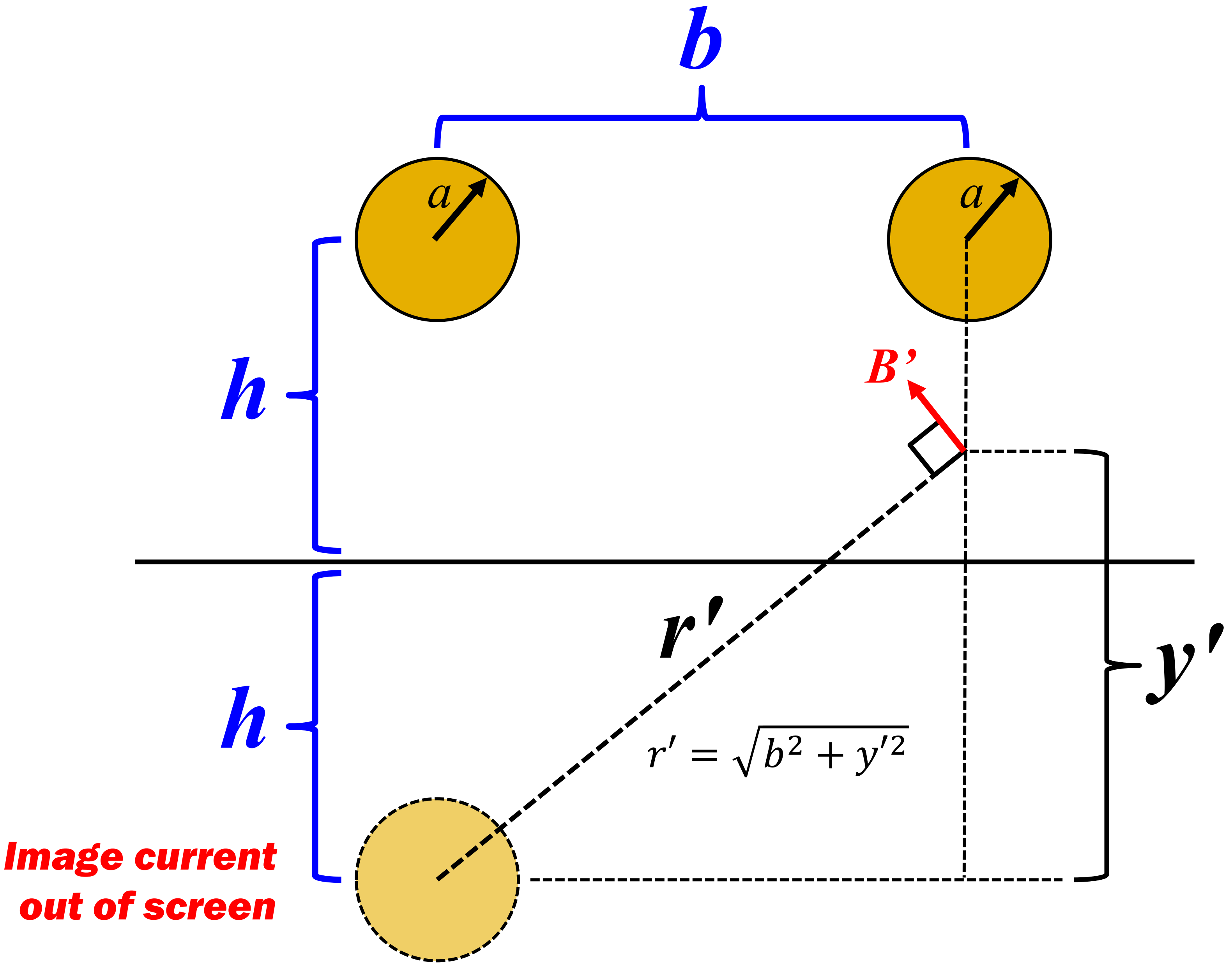
Total flux from culprit wire:

$$\Phi = \frac{\mu I l}{2\pi} \int_a^h \frac{y\,dy}{(b^2 + y^2)}$$

Flux per unit length from culprit wire:

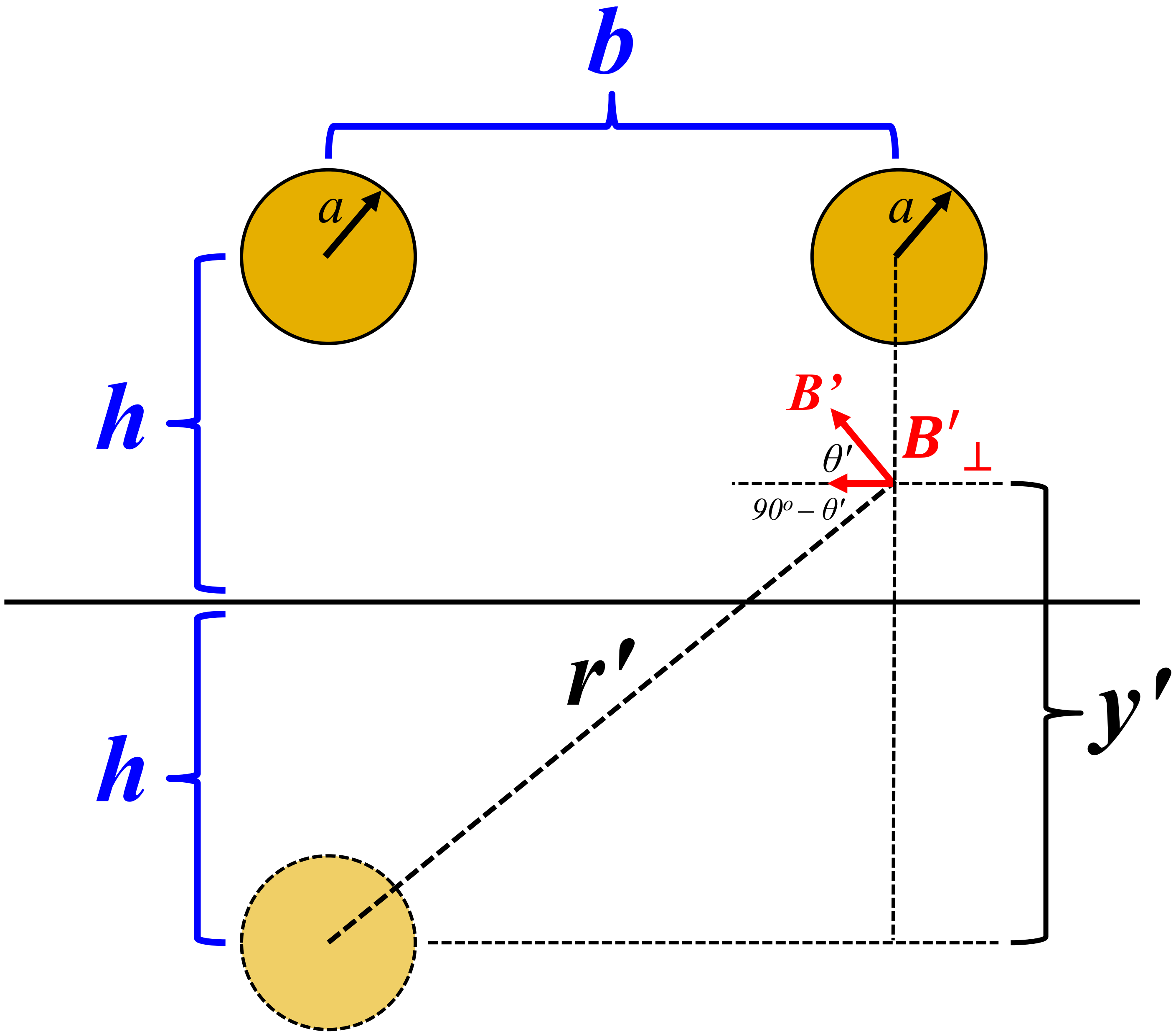
$$\Phi_l = \frac{\mu I}{2\pi} \int_a^h \frac{y\,dy}{(b^2 + y^2)}$$


b
a
a
h
h
a
Image current
out of screen



$$B' = \frac{\mu I}{2\pi r'}$$

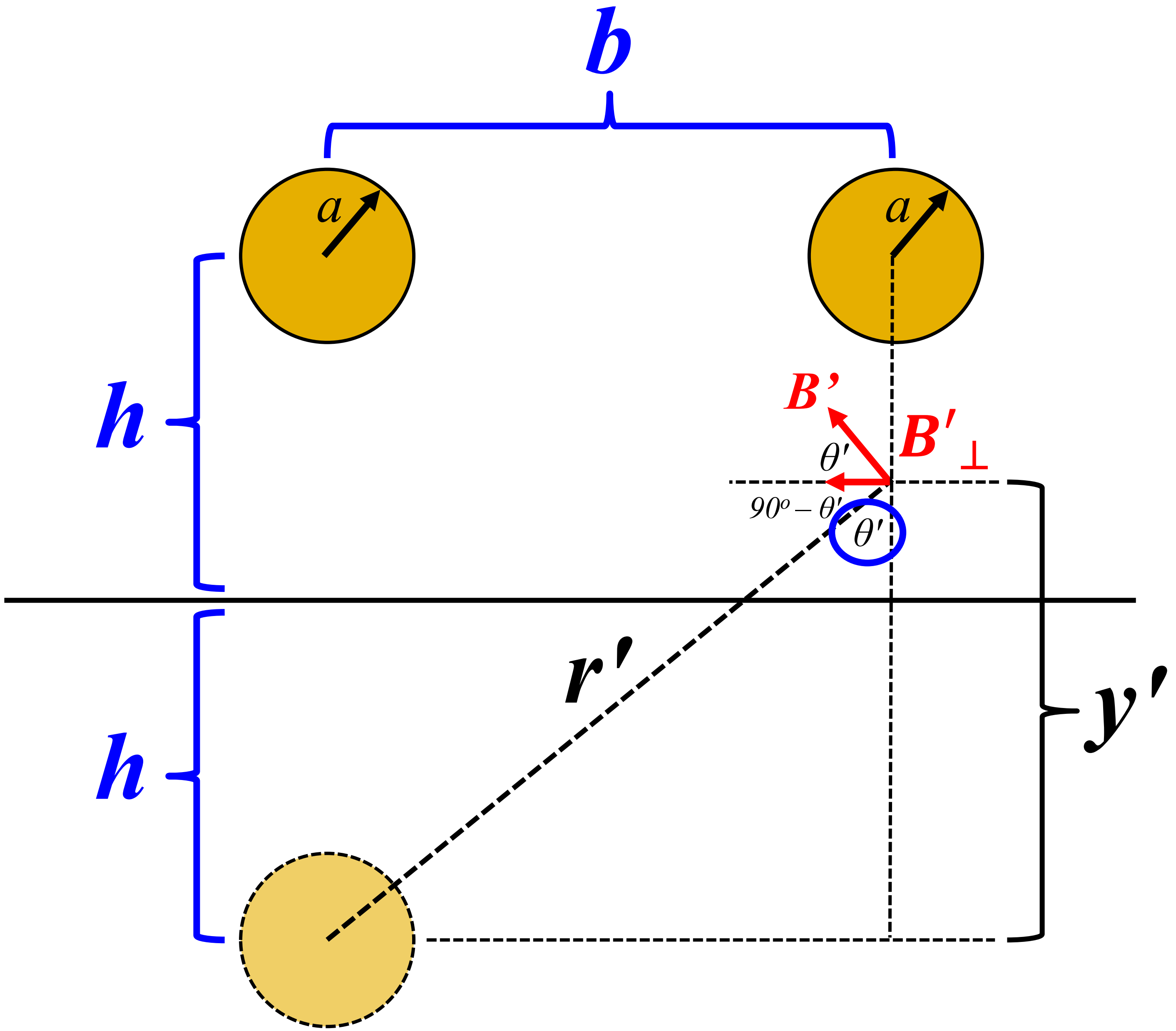
$$B' = \frac{\mu I}{2\pi\sqrt{b^2 + y'^2}}$$



$$B' = \frac{\mu I}{2\pi r'}$$

$$B' = \frac{\mu I}{2\pi\sqrt{b^2 + y'^2}}$$

$$B'_{\perp} = B' \cos\theta'$$



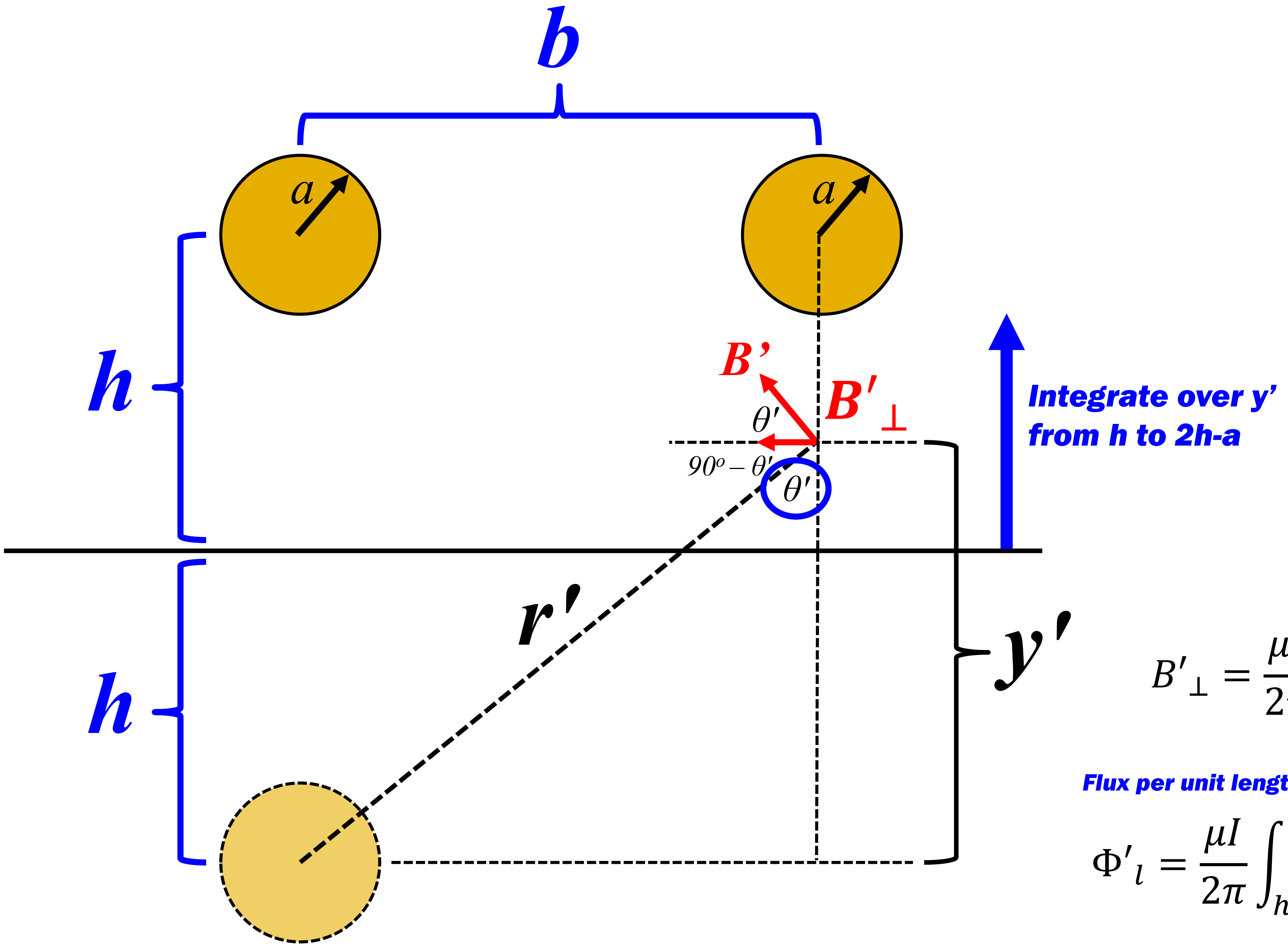
$$B' = \frac{\mu I}{2\pi r'}$$

$$B' = \frac{\mu I}{2\pi\sqrt{b^2 + y'^2}}$$

$$B'_\perp = B' \cos\theta'$$

$$\cos\theta' = \frac{y'}{\sqrt{b^2 + y'^2}}$$

$$B'_\perp = \frac{\mu I}{2\pi} \cdot \frac{y'}{(b^2 + y'^2)}$$



$$B'_\perp = \frac{\mu I}{2\pi} \cdot \frac{y'}{(b^2 + y'^2)}$$

Flux per unit length from image wire:

$$\Phi'_l = \frac{\mu I}{2\pi} \int_h^{2h-a} \frac{y'\,dy'}{(b^2 + y'^2)}$$

Parallel Wires Above Ground Plane (cont.)

Total flux per unit length from culprit and image wires:

$$\Phi_{\mathrm{TOT}} = \Phi_{\mathrm{l}} + \Phi'_{\mathrm{l}}$$

$$\Phi_{\mathrm{TOT}} = \frac{\mu I}{2\pi}\int_a^h \frac{y\,dy}{(b^2+y^2)} + \frac{\mu I}{2\pi}\int_h^{2h-a} \frac{y'\,dy'}{(b^2+y'^2)}$$

$$\Phi_{\mathrm{TOT}} = \frac{\mu I}{4\pi}\cdot\left[\ln(b^2+y^2)\Big|_a^h + \ln(b^2+y'^2)\Big|_h^{2h-a}\right]$$

$$\Phi_{\mathrm{TOT}} = \frac{\mu I}{4\pi}\cdot\underbrace{\left[\ln\left(\frac{b^2+h^2}{b^2+a^2}\right) + \ln\left(\frac{b^2+(2h-a)^2}{b^2+h^2}\right)\right]}_{\ln(xy)\,=\,\ln x+\ln y}$$

$$\downarrow$$

$$\Phi_{\mathrm{TOT}} = \frac{\mu I}{4\pi}\cdot\ln\left[\frac{b^2+(2h-a)^2}{b^2+a^2}\right]$$

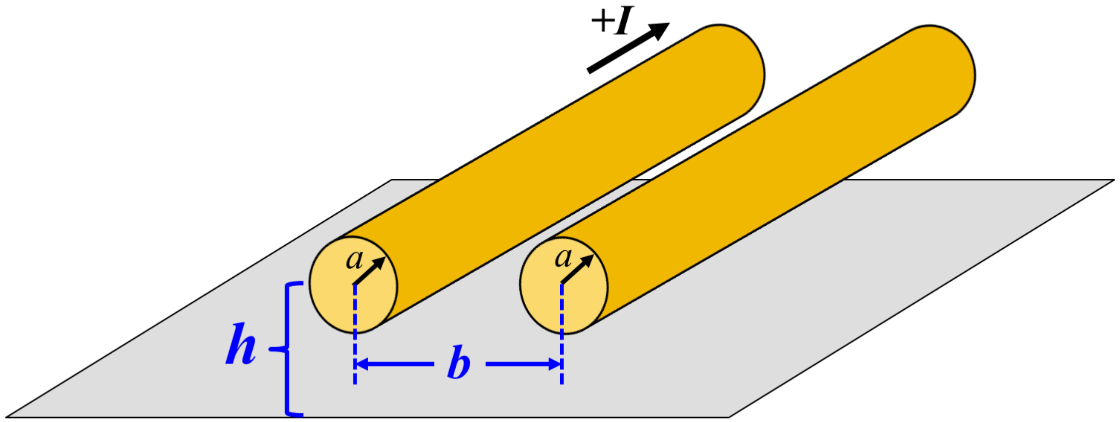
$$L_M = \frac{\mu}{4\pi}\cdot\ln\left[\frac{b^2+(2h-a)^2}{b^2+a^2}\right]$$

$$\downarrow$$

$$L_M = \frac{\mu}{4\pi}\cdot\ln\left[\frac{\left(\frac{b}{a}\right)^2+\left(\frac{2h}{a}-1\right)^2}{\left(\frac{b}{a}\right)^2+1}\right]$$

Mutual inductance per unit length for parallel wires above ground plane

Parallel Wires Above Ground Plane (cont.)



$$L_M = \frac{\mu}{4\pi} \cdot \ln\left[\frac{\left(\frac{b}{a}\right)^2 + \left(\frac{2h}{a} - 1\right)^2}{\left(\frac{b}{a}\right)^2 + 1}\right]$$

$$\mu = \mu_r \mu_0 \qquad \mu_r = 1$$